Fourier Series

Elements of Solution

Problem 1. Let $f : \mathbf{R} \longrightarrow \mathbf{C}$ be a 2π-periodic function, piecewise continuous, piecewise of class C^1. For $x_0 \in \mathbf{R}$, we denote by $f(x_0^{\pm})$ the one-sided limit $\lim\limits_{x \to x_0^{\pm}} f(x)$ and $\tilde{f}$ is the function defined on $\mathbf{R}$ by

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}.$$

The purpose of this problem is to establish the pointwise convergence of the Fourier series of f to $\tilde{f}$, that is, for any $x_0 \in \mathbf{R}$,

$$\sum_{n \in \mathbf{Z}} \hat{f}(n) e^{inx_0} = \tilde{f}(x_0).$$

(a) Verify that for any x_0 in $\mathbf{R}$, the map

$$h \mapsto \frac{f(x_0 + h) + f(x_0 - h) - f(x_0^+) - f(x_0^-)}{h}$$

is bounded near 0.

First, we consider the case $x_0 = 0$. Denote by $S_N(f)(0)$ the partial sum $\sum\limits_{n=-N}^{N} \hat{f}(n)$.

(b) Prove that

$$2\pi S_N(f)(0) = \int_0^{\pi} (f(x) + f(-x)) D_N(x)\, dx,$$

where $D_N(x)$ is the Dirichlet kernel $\dfrac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$.

(c) Show that $2\pi\big(S_N(f)(0) - \tilde{f}(0)\big)$ can be written as $\displaystyle\int_0^{\pi} g(x) \sin\left(N + \frac{1}{2}\right) x\, dx$ with g piecewise continuous and bounded near 0.

(d) Conclude and extend to the case of arbitrary x_0.

From now on, we assume f continuous and piecewise of class C^1. We denote by φ the function defined on $\mathbf{R}$ by

$$\varphi(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x, \\[2ex] \dfrac{f'(x^+) + f'(x^-)}{2} & \text{otherwise.} \end{cases}$$

(e) Verify the relation $\hat{\varphi}(n) = in\,\hat{f}(n)$ for all $n \in \mathbf{Z}$.

(f) Prove that the Fourier series of f converges normally to $\tilde{f}$.

<u>*Hints*</u>*:* **(d)** Riemann-Lebesgue. Consider $f_{x_0} : x \mapsto f(x + x_0)$. **(f)** $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

Solution. **(a)** Boundedness follows from the existence of limits on the the left and the right for the function and its derivative.

(b) Partial sums of Fourier series are given by right convolution with Dirichlet's kernel, which is an even function.

(c) The function $g(x) = \dfrac{f(x) + f(-x) - f(0^+) - f(0^-)}{\sin\left(\frac{x}{2}\right)}$ is bounded near 0 by the hypotheses and the fact that $\sin(x) \sim_0 x$.

(d) The integral converges to 0 as $N \to \infty$ by the Riemann-Lebesgue Lemma. For the general case, observe that $\widehat{f_{x_0}}(n) = e^{inx_0}\hat{f}(n)$.

(e) Integrate by parts on every interval where the function is of class C^1.

(f) For every n, we have $|\hat{f}(n)| = \left|\dfrac{\hat{\varphi}(n)}{n}\right| \leq \dfrac{1}{2}\left(|\hat{\varphi}(n)|^2 + \dfrac{1}{n^2}\right)$, summable by Parseval. Therefore, the series converges normally to its pointwise limit $\tilde{f}$.

Problem 2. Let f be the 2π-periodic function on $\mathbf{R}$ defined by $f(x) = 1 - \dfrac{x^2}{\pi^2}$ for all $x \in [-\pi, \pi]$.

1. Compute the Fourier coefficients of f.
2. Deduce the sums of the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$, $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ and $\displaystyle\sum_{n\geq 1} \frac{1}{n^4}$.

Hints: Note that only the real part of $\hat{f}(n)$ is useful. Parseval.

Solution. A direct computation shows that $\hat{f}(0) = \dfrac{2}{3}$ and that the real part of $\hat{f}(n)$ is

$$\Re\big(\hat{f}(n)\big) = \frac{2(-1)^{n+1}}{\pi^2 n^2}.$$

Since f clearly satisfies the hypotheses of the results proved in the previous problem, we get:

- $f(\pi) = 0 = \dfrac{2}{3} - \dfrac{2}{\pi^2} \displaystyle\sum_{|n|\geq 1} \frac{1}{n^2}$ so that $\displaystyle\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.
- $f(0) = 1 = \dfrac{2}{3} - \dfrac{2}{\pi^2} \displaystyle\sum_{|n|\geq 1} \frac{(-1)^n}{n^2}$ so that $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.

Finally, Parsevals' Identity gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{x^2}{\pi^2}\right)^2 dx = \frac{8}{15} = \frac{4}{9} + \frac{4}{\pi^4} \sum_{|n|\geq 1} \frac{1}{n^4}$$

so that $\displaystyle\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.

Problem 3. Let $\mathcal{C}_{2\pi}$ denote the space of 2π-periodic continuous functions on $\mathbf{R}$, equipped with $\|\cdot\|_\infty$. For $N \in \mathbf{N}$, we define a linear functional φ_N on $\mathcal{C}_{2\pi}$ by:

$$\varphi_N(f) = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n).$$

(a) Verify that $\mathcal{C}_{2\pi}$ is a Banach space.

(b) Prove that $\varphi_N \in \mathcal{C}_{2\pi}^*$ and compute $\|\varphi_N\|$.

(c) Show that $\|\varphi_N\| \geq \dfrac{2}{\pi}\displaystyle\int_0^{\frac{(2N+1)\pi}{2}} \left|\frac{\sin u}{u}\right| du$ for any $N \in \mathbf{N}$.

(d) Prove the existence of a function in $\mathcal{C}_{2\pi}$ whose Fourier series diverges at 0.

Hints: **(b)** Consider $f_\varepsilon = \dfrac{D_N}{|D_N| + \varepsilon}$. **(d)** Use the Principle of Uniform Boundedness.

Solution. **(a)** The space $\mathcal{C}_{2\pi}$ is a closed subspace of the Banach space of bounded functions on $\mathbf{R}$.

(b) Using the Dirichlet kernel once more, we see that $\varphi_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(x)\,dx$ from which it follows that $\|\varphi_N\| \leq \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |D_N(x)|\,dx$, so that $\varphi_N \in \mathcal{C}_{2\pi}^*$. To prove the reverse inequality, consider $f_\varepsilon = \frac{D_N}{|D_N|+\varepsilon}$ for $\varepsilon > 0$. It is clearly in the unit ball and $\displaystyle\lim_{\varepsilon\to 0} \varphi_N(f_\varepsilon) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx$ so finally,

$$\|\varphi_N\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx.$$

(c) It follows from the inequality $\left|\sin\left(\dfrac{x}{2}\right)\right| \leq \left|\dfrac{x}{2}\right|$ and the change of variables $u = \left(N + \dfrac{1}{2}\right)x$.

(d) The improper integral $\displaystyle\int_0^{\infty} \left|\frac{\sin u}{u}\right| du$ is divergent so $\displaystyle\lim_{N\to\infty} \|\varphi_N\| = \infty$. If $\varphi_N(f)$ was convergent for all $f \in \mathcal{C}_{2\pi}$, the Principle of Uniform Boundedness would imply that $\|\varphi_N\|$ is a bounded sequence, so there exist functions whose Fourier series must diverge at 0.

Note that such functions can be explicitly constructed, see for instance Chapter 3 in [Stein - Shakarchi].