Fourier Series

Due Apr. 30

Problem 1. Let $f : \mathbf{R} \longrightarrow \mathbf{C}$ be a 2π-periodic function, piecewise continuous, piecewise of class C^1. For $x_0 \in \mathbf{R}$, we denote by $f(x_0^{\pm})$ the one-sided limit $\lim\limits_{x \to x_0^{\pm}} f(x)$ and $\tilde{f}$ is the function defined on $\mathbf{R}$ by

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}.$$

The purpose of this problem is to establish the pointwise convergence of the Fourier series of f to $\tilde{f}$, that is, for any $x_0 \in \mathbf{R}$,

$$\sum_{n \in \mathbf{Z}} \hat{f}(n) e^{inx_0} = \tilde{f}(x_0).$$

(a) Verify that for any x_0 in $\mathbf{R}$, the map

$$h \mapsto \frac{f(x_0 + h) + f(x_0 - h) - f(x_0^+) - f(x_0^-)}{h}$$

is bounded near 0.

First, we consider the case $x_0 = 0$. Denote by $S_N(f)(0)$ the partial sum $\sum\limits_{n=-N}^{N} \hat{f}(n)$.

(b) Prove that

$$2\pi S_N(f)(0) = \int_0^{\pi} (f(x) + f(-x)) D_N(x) \, dx,$$

where $D_N(x)$ is the Dirichlet kernel $\dfrac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$.

(c) Show that $2\pi\big(S_N(f)(0) - \tilde{f}(0)\big)$ can be written as $\displaystyle\int_0^{\pi} g(x) \sin\left(N + \frac{1}{2}\right) x \, dx$ with g piecewise continuous and bounded near 0.

(d) Conclude and extend to the case of arbitrary x_0.

From now on, we assume f continuous and piecewise of class C^1. We denote by φ the function defined on $\mathbf{R}$ by

$$\varphi(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x, \\ \dfrac{f'(x^+) + f'(x^-)}{2} & \text{otherwise.} \end{cases}$$

(e) Verify the relation $\hat{\varphi}(n) = in\,\hat{f}(n)$ for all $n \in \mathbf{Z}$.

(f) Prove that the Fourier series of f converges normally to $\tilde{f}$.

<u>Hints</u>: **(d)** Riemann-Lebesgue. Consider $f_{x_0} : x \mapsto f(x + x_0)$. **(f)** $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

Problem 2. Let f be the 2π-periodic function on $\mathbf{R}$ defined by $f(x) = 1 - \dfrac{x^2}{\pi^2}$ for all $x \in [-\pi, \pi]$.

1. Compute the Fourier coefficients of f.
2. Deduce the sums of the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$, $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ and $\displaystyle\sum_{n\geq 1} \frac{1}{n^4}$.

<u>Hints</u>: Note that only the real part of $\hat{f}(n)$ is useful. Parseval.

Problem 3. Let $\mathcal{C}_{2\pi}$ denote the space of 2π-periodic continuous functions on $\mathbf{R}$, equipped with $\|\cdot\|_\infty$. For $N \in \mathbf{N}$, we define a linear functional φ_N on $\mathcal{C}_{2\pi}$ by:

$$\varphi_N(f) = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n).$$

(a) Verify that $\mathcal{C}_{2\pi}$ is a Banach space.

(b) Prove that $\varphi_N \in \mathcal{C}_{2\pi}^*$ and compute $\|\varphi_N\|$.

(c) Show that $\|\varphi_N\| \geq \dfrac{2}{\pi} \displaystyle\int_0^{\frac{(2N+1)\pi}{2}} \left|\frac{\sin u}{u}\right| du$ for any $N \in \mathbf{N}$.

(d) Prove the existence of a function in $\mathcal{C}_{2\pi}$ whose Fourier series diverges at 0.

Hints: **(b)** Consider $f_\varepsilon = \dfrac{D_N}{|D_N| + \varepsilon}$. **(d)** Use the Principle of Uniform Boundedness.