Duality

Elements of Solution

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1. (*Weak boundedness.*) Let E and F be Banach spaces and $T : E \longrightarrow F$ a linear map such that

$$\forall \varphi \in F^* \quad , \quad \varphi \circ T \in E^*.$$

Prove that T is bounded.

Solution. By the Closed Graph Theorem, it is enough to consider a sequence $\{(x_n, Tx_n)\}_{n\in\mathbf{N}}$ that converges to (x, y) in $E \times F$ and to prove that $Tx = y$. Let $\varphi \in F^*$. The continuity of φ and that of $\varphi \circ T$ imply that $\varphi(Tx) = \varphi(y)$ for all $\varphi \in F^*$. By Hahn-Banach, this implies that $Tx = y$.

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2. (*Dual of* $\ell^p(\mathbf{N})$.) Let $p \in [1, +\infty)$ and denote by q the only element of $(1, +\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The purpose of this problem is to identify $\ell^p(\mathbf{N})^*$ with $\ell^q(\mathbf{N})$. To this end, we shall prove that the map Φ defined on ℓ^q by

$$\Phi(u)v = \sum_{n\geq 0} u_n v_n$$

is an isometry.

(a) Recall Hölder's Inequality and verify that $\Phi(u)$ is a linear functional on $\ell^p(\mathbf{N})$ for each $u \in \ell^q(\mathbf{N})$ and that Φ is linear.

(b) Let $u \in \ell^q(\mathbf{N})$. Prove that $\Phi(u) \in \ell^p(\mathbf{N})^*$ and that $\|\Phi(u)\| \leq \|u\|_q$.

Let $u \in \ell^q(\mathbf{N})$ be fixed.

(c) Assume $p > 1$. Verify that the sequence v defined by

$$v_n = \|u\|_q^{1-q} \operatorname{sign}(u_n) \, |u_n|^{q-1}$$

is in ℓ^p and compute $\Phi(u)v$.

(d) Let $p = 1$. For $\varepsilon > 0$, find v on the unit sphere of $\ell^1(\mathbf{N})$ such that

$$|\Phi(u)v| > \|u\|_\infty - \varepsilon.$$

(e) What have we proved so far?

(f) Prove that finitely supported sequences are dense in $\ell^p(\mathbf{N})$ for $p \geq 1$.

(g) Does the result hold in $\ell^\infty(\mathbf{N})$?

For $n \in \mathbf{N}$, define the sequence e^n by $e^n_k = \delta_{k,n}$, that is

$$e^n = \{\overbrace{0, 0, \ldots, 0}^{n}, 1, 0, 0, \ldots\}.$$

Let $\varphi \in \ell^p(\mathbf{N})^*$ and $\gamma_n = \varphi(e^n)$. For $N \in \mathbf{N}$, define a sequence δ^N by

$$\delta^N = \{\gamma_0|\gamma_0|^{q-2}, \gamma_1|\gamma_1|^{q-2}, \ldots, \gamma_N|\gamma_N|^{q-2}, 0, 0, \ldots\}.$$

(h) Calculate $\varphi(\delta^N)$.

(i) Prove that $\displaystyle\sum_{n=0}^{N} |\gamma_n|^q \leq \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^q\right)^{\frac{1}{p}}$.

(j) Deduce that the N-truncation of the sequence $\gamma = \{\gamma_n\}_{n\in\mathbf{N}}$ has norm less than $\|\varphi\|$ in $\ell^q(\mathbf{N})$.

(k) Conclude that γ is in $\ell^q(\mathbf{N})$.

(l) Verify that $\varphi(u) = \Phi(\gamma)(u)$ if u is finitely supported and conclude.

(m) Prove the existence of a bounded linear functional on $\ell^\infty(\mathbf{N})$ that is not of the form $\Phi(u)$ with $u \in \ell^1(\mathbf{N})$.

Hint: consider the subspace C of convergent sequences and study the map:

$$\lambda : v \mapsto \lim_{n\to\infty} v_n.$$

Solution. **(a)**

(b)

(c) Note that $\dfrac{1}{q-1} = \dfrac{p}{q}$. Direct computations show that $\|v\|_p = 1$ and that $\Phi(u)v = \|u\|_q$.

(d) Let $n_0 \in \mathbf{N}$ be such that $|u_{n_0}| > \|u\|_\infty - \varepsilon$ and define v by $v_n = \delta_{n,n_0}$.

(e) For each $u \in \ell^q(\mathbf{N})$, the linear functional $\Phi(u)$ is bounded on $\ell^p(\mathbf{N})$ and its operator norm is $\|u\|_q$. Therefore, Φ is an isometric embedding of $\ell^q(\mathbf{N})$ into $\ell^p(\mathbf{N})^*$ and all is left to prove is surjectivity.

(f) If p is finite, every sequence in $\ell^p(\mathbf{N})$ is the limit of its truncations.

(g) No: non-zero constant sequences are not $\|\cdot\|_\infty$-limits of finitely supported sequences.

(h) Observe that $\delta^N = \sum_{n=0}^{N} \gamma_n |\gamma_n|^{q-2}\, e^n$ so by linearity,

$$\varphi(\delta^N) = \sum_{n=0}^{N} \gamma_n |\gamma_n|^{q-2} \underbrace{\varphi(e^n)}_{=\gamma_n} = \sum_{n=0}^{N} |\gamma_n|^q.$$

(i) Since φ is continuous, the inequality $|\varphi(\delta^N)| \le \|\varphi\|\, \|\delta^N\|_p$ becomes

$$\sum_{n=0}^{N} |\gamma_n|^q \le \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^{(q-1)p} \right)^{\frac{1}{p}} = \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^q \right)^{\frac{1}{p}}.$$

(j) If γ^N denote the N-truncation of γ, the previous inequality reads

$$\|\gamma^N\|_q^q \le \|\varphi\|\, \|\gamma^N\|_q^{\frac{q}{p}}$$

that is, $\|\gamma^N\|_q^{q-\frac{q}{p}} \le \|\varphi\|$. Note that $q - \dfrac{q}{p} = 1$ to conclude.

(k) The bound on $\|\gamma^N\|_q$ is independent of the order of the truncation N so, letting $N \to \infty$, we see that $\|\gamma\|_q \le \|\varphi\| < \infty$.

(l) The identity $\varphi(u) = \Phi(\gamma)(u)$ for finitely supported u is a direct consequence of the definition of γ. Since φ and $\Phi(\gamma)$ are continuous and coincide on a dense subspace, they must be equal, so φ is in the range of Φ, which is therefore surjective. The argument (and the result) fail for $p = \infty$ as we shall see in the next question.

(m) Consider the subspace C of convergent sequences and define

$$\lambda : v \mapsto \lim_{n\to\infty} v_n.$$

Observe that the linear functional λ is continuous on C with $\|\lambda\| = 1$. By Hahn-Banach, it extends to a bounded linear functional Λ on $\ell^\infty(\mathbf{N})$, with $\|\Lambda\| = 1$. Assume that Λ can be represented by $u \in \ell^1(\mathbf{N})$, that is $\Lambda(v) = \Phi(u)v$ for all $v \in \ell^\infty(\mathbf{N})$. Then, with the notation of question 5., we get $u_n = \Phi(u)e^n = \Lambda(e^n) = \lambda(e^n) = 0$ for all $n \ge 0$, so $u = 0$ hence $\lambda = 0$, which is excluded since $\|\lambda\| = 1$.

3. (*Closed convex sets that cannot be separated.*) Let E_0 and F be the subsets of $\ell^1(\mathbf{N})$ defined by

$$E_0 = \left\{u \in \ell^1(\mathbf{N}) \, , \, \forall n \geq 0 \, , \, u_{2n} = 0\right\}$$

and

$$F = \left\{u \in \ell^1(\mathbf{N}) \, , \, \forall n \geq 1 \, , \, u_{2n} = 2^{-n} u_{2n-1}\right\} .$$

(a) Verify that E_0 and F are closed subspaces and that $\overline{E_0 + F} = \ell^1(\mathbf{N})$.

Hint: show that E_0 and F are intersections of closed hyperplanes.

Let v be the sequence defined by $v_{2n} = 2^{-n}$ and $v_{2n-1} = 0$.

(b) Verify that v is in $\ell^1(\mathbf{N})$ and that $v \notin E_0 + F$.

(c) Let $E = E_0 - v$. Prove that E and F are closed disjoint convex subsets of $\ell^1(\mathbf{N})$ that cannot be separated in the sense that there exists no couple (φ, α) in $\ell^1(\mathbf{N})^* \times \mathbf{R}$ such that $\varphi \neq 0$ and

$$\varphi(e) \leq \alpha \leq \varphi(f)$$

for all $e \in E$, $f \in F$.

Solution. **(a)** Both spaces can be written as intersections of kernels of bounded linear functionals so they are closed. To prove the density of the sum, it suffices to prove that it contains all finitely supported sequences. Let u be a sequence supported on $\{0, 1, \ldots, p\}$, that is, $u_n = 0$ for $n > p$. Then, define e and f by

$$\begin{cases} e_{2n} = 0 & e_{2n-1} = u_{2n-1} - 2^n u_{2n} \\ f_{2n} = u_{2n} & f_{2n-1} = 2^n u_{2n} \end{cases}$$

and observe that $u = e + f$.

(b) Assume that $v = e + f$ with $e \in E_0$ and $f \in F$. Then, $e_{2n-1} = -1$ and $f_{2n-1} = 1$ for all $n \geq 1$ which contradicts e and f being in $\ell^1(\mathbf{N})$.

(c) If there was a separating couple (φ, α), then φ would be bounded above by $\alpha + \varphi(v)$ on E_0 and bounded below by α on F. Since they are linear subspaces, it means that φ would vanish on both hence on the sum. Since the latter is dense in $\ell^1(\mathbf{N})$ and φ assumed continuous, this implies $\varphi = 0$.