Duality

Due Apr. 12

Problem 1. Let E and F be Banach spaces and $T : E \longrightarrow F$ a linear map such that

$$\forall \varphi \in F^* \quad , \quad \varphi \circ T \in E^*.$$

Prove that T is bounded.

Problem 2. Let $p \in [1, +\infty)$ and denote by q the only element of $(1, +\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The purpose of this problem is to identify $\ell^p(\mathbf{N})^*$ with $\ell^q(\mathbf{N})$. To this end, we shall prove that the map Φ defined on ℓ^q by

$$\Phi(u)v = \sum_{n \geq 0} u_n v_n$$

is an isometry.

(a) Recall Hölder's Inequality and verify that $\Phi(u)$ is a linear functional on $\ell^p(\mathbf{N})$ for each $u \in \ell^q(\mathbf{N})$ and that Φ is linear.

(b) Let $u \in \ell^q(\mathbf{N})$. Prove that $\Phi(u) \in \ell^p(\mathbf{N})^*$ and that $\|\Phi(u)\| \leq \|u\|_q$.

Let $u \in \ell^q(\mathbf{N})$ be fixed.

(c) Assume $p > 1$. Verify that the sequence v defined by

$$v_n = \|u\|_q^{1-q} \operatorname{sign}(u_n) \, |u_n|^{q-1}$$

is in ℓ^p and compute $\Phi(u)v$.

(d) Let $p = 1$. For $\varepsilon > 0$, find v on the unit sphere of $\ell^1(\mathbf{N})$ such that

$$|\Phi(u)v| > \|u\|_\infty - \varepsilon.$$

(e) What have we proved so far?

(f) Prove that finitely supported sequences are dense in $\ell^p(\mathbf{N})$ for $p \geq 1$.

(g) Does the result hold in $\ell^\infty(\mathbf{N})$?

For $n \in \mathbf{N}$, define the sequence e^n by $e^n_k = \delta_{k,n}$, that is

$$e^n = \{\overbrace{0, 0, \dots, 0}^{n}, 1, 0, 0, \dots\}.$$

Let $\varphi \in \ell^p(\mathbf{N})^*$ and $\gamma_n = \varphi(e^n)$. For $N \in \mathbf{N}$, define a sequence δ^N by

$$\delta^N = \{\gamma_0|\gamma_0|^{q-2}, \gamma_1|\gamma_1|^{q-2}, \dots, \gamma_N|\gamma_N|^{q-2}, 0, 0, \dots\}.$$

(h) Calculate $\varphi(\delta^N)$.

(i) Prove that $\displaystyle\sum_{n=0}^{N} |\gamma_n|^q \leq \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^q\right)^{\frac{1}{p}}$.

(j) Deduce that the N-truncation of the sequence $\gamma = \{\gamma_n\}_{n\in\mathbf{N}}$ has norm less than $\|\varphi\|$ in $\ell^q(\mathbf{N})$.

(k) Conclude that γ is in $\ell^q(\mathbf{N})$.

(l) Verify that $\varphi(u) = \Phi(\gamma)(u)$ if u is finitely supported and conclude.

(m) Prove the existence of a bounded linear functional on $\ell^\infty(\mathbf{N})$ that is not of the form $\Phi(u)$ with $u \in \ell^1(\mathbf{N})$.

Hint: consider the subspace C of convergent sequences and study the map:

$$\lambda : v \mapsto \lim_{n\to\infty} v_n.$$

Problem 3. Let E_0 and F be the subsets of $\ell^1(\mathbf{N})$ defined by

$$E_0 = \left\{u \in \ell^1(\mathbf{N})\,,\, \forall n \geq 0\,,\, u_{2n} = 0\right\}$$

and

$$F = \left\{u \in \ell^1(\mathbf{N})\,,\, \forall n \geq 1\,,\, u_{2n} = 2^{-n} u_{2n-1}\right\}.$$

(a) Verify that E_0 and F are closed subspaces and that $\overline{E_0 + F} = \ell^1(\mathbf{N})$.

Hint: show that E_0 and F are intersections of closed hyperplanes.

Let v be the sequence defined by $v_{2n} = 2^{-n}$ and $v_{2n-1} = 0$.

(b) Verify that v is in $\ell^1(\mathbf{N})$ and that $v \notin E_0 + F$.

(c) Let $E = E_0 - v$. Prove that E and F are closed disjoint convex subsets of $\ell^1(\mathbf{N})$ that cannot be separated in the sense that there exists no couple (φ, α) in $\ell^1(\mathbf{N})^* \times \mathbf{R}$ such that $\varphi \neq 0$ and

$$\varphi(e) \leq \alpha \leq \varphi(f)$$

for all $e \in E$, $f \in F$.