Linear Operators on Banach Spaces

Elements of Solution

---

1. (*Integration.*) Consider, for $x_0 \in [a,b]$, the linear map $\mathcal{I}_{x_0} : C([a,b]) \longrightarrow C([a,b])$ defined by

$$\mathcal{I}_{x_0} f(x) = \int_{x_0}^{x} f(t)\, dt.$$

(a) Prove that $\mathcal{I}_{x_0}$ is a bounded operator and that $\|\mathcal{I}_{x_0}\|_{\mathrm{op}} \leq b - a$.

(b) Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of functions with continuous derivatives such that:

- the sequence $\{f_n'\}_{n\in\mathbf{N}}$ converges uniformly on $[a,b]$;
- there exists a point $x_0 \in [a,b]$ such that $\{f_n(x_0)\}_{n\in\mathbf{N}}$ converges.

Prove that $\{f_n\}_{n\in\mathbf{N}}$ converges in $C([a,b])$.

Solution. **(a)** It suffices to show that $b-a$ is a Lipschitz constant for $\mathcal{I}_{x_0}$. Denote by $I(x)$ the interval with bounds x_0 and x. Since

$$|\mathcal{I}_{x_0} f(x)| \leq \int_{I(x)} |f(t)|\ dt \leq |x - x_0|\ \|f\|_\infty \leq (b-a)\|f\|_\infty$$

holds for any $x \in [a,b]$, we see that $\|\mathcal{I}_{x_0} f\|_\infty \leq (b-a) \leq \|f\|_\infty$.

(b) Denote by g the (uniform) limit of the sequence $\{f_n'\}_{n\in\mathbf{N}}$ and $\ell = \lim_n f_n(x_0)$. Then $\ell + \mathcal{I}_{x_0}(g)$ is a continuous function. The Fundamental Theorem of Calculus gives

$$f_n(x) = f_n(x_0) + \mathcal{I}_{x_0} g(x)$$

for any $x \in [a,b]$, so that

$$|f_n(x) - f(x)| \leq |f_n(x_0) - \ell| + |\mathcal{I}_{x_0}(f_n' - g)| \leq |f_n(x_0) - \ell| + (b-a)\|f_n' - g\|_\infty$$

and both terms in the right-hand side can be made arbitrarily small for n large enough, independently of x.

---

2. (*Differentiation.*) Denote by E the Banach space $C([0,1])$ equipped with supremum norm and let $E^1 = C^1([0,1])$ be the subspace of all functions that admit a continuous derivative.

The purpose of this problem is to study properties of the differentiation operator

$$D : \begin{array}{rcl} E^1 & \longrightarrow & E \\ f & \longmapsto & f' \end{array} .$$

(a) Is D bounded?

(b) Is D closed?

(c) Is E^1 closed in E? If not, determine its closure.

Let F be a closed subspace of E, contained in E^1.

(d) Prove that the restriction of D to F is Lipschitz.

(e) Prove that F is necessarily finite-dimensional.

Solution. **(a)** No: let f be a function of class C^1 on $\mathbf{R}$ such that $f'(0) \neq 0$ for some $x_0 \in [0,1]$ and define for $n \in \mathbf{N}$, $f_n(x) = f(nx)$ for $x \in [0,1]$. The sequence $\{f_n\}_{n\in\mathbf{N}}$ is bounded but

$$\|Df_n\|_\infty \geq f_n'(0) = n|f'(0)|$$

so Df_n is unbounded.

(b) Yes, as a direct consequence of the result established in the previous problem.

(c) The space E^1 contains all polynomial functions, and is therefore dense in E by Weierstrass' Theorem. Being a strict subspace, it is in particular not closed.

(d) The same argument as in **(b)** shows that D is closed. Since F is closed in E, it is complete and the Closed Graph Theorem applies, implying that D is bounded on F, hence Lipschitz.

(e) By Riesz' Theorem, it suffices to prove that the closed unit ball $\mathbb{B}_1$ in F is compact. If C be a Lipschitz constant for D, then

$$\|f'\|_\infty \leq C$$

for every f in $\mathbb{B}_1$. The Mean Value Theorem implies that $\mathbb{B}_1$ is equicontinuous, hence compact by Ascoli-Arzelà.

---

3. (*Banach-Steinhaus' Uniform Boundedness Principle.*) Let E be a Banach space, F a normed linear space and $\mathcal{T}$ a family of bounded operators between E and F. The purpose of this problem is to prove the equivalence of the following statements:

(i) $\mathcal{T}$ is bounded in $\mathcal{B}(E, F)$.

(ii) For every x in E, the family $\mathcal{T}_x = \{Tx \,:\, T \in \mathcal{T}\}$ is bounded in F.

(a) Check that (i) implies (ii).

For $n \in \mathbf{N}$, let
$$E_n = \{x \in E \,:\, \|Tx\| \leq n \quad \text{for all } T \text{ in } \mathcal{T}\}.$$

(b) Verify that the sets E_n are closed and that $E = \bigcup_{n\in\mathbf{N}} E_n$.

(c) Prove the existence of a point $x_0 \in E$, a real number $r > 0$ and an integer n_0 such that
$$\forall T \in \mathcal{T}\,,\ \|x - x_0\| < r \quad \Rightarrow \quad \|Tx\| \leq n_0.$$

(d) Find a common Lipschitz constant for all T in $\mathcal{T}$ and conclude.

Solution. **(a)** If $\mathcal{T}$ is bounded, there exists $M \geq 0$ such that $\|T\|_{\text{op}} \leq M$ for any $T \in \mathcal{T}$. Since M is a Lipschitz constant for every T, it follows that $\mathcal{T}_x$ is contained in the closed ball centered at 0, with radius $M\|x\|$.

(b) The sets E_n are closed as intersections of closed sets, namely the inverse images of $B_c(0_F, n)$ under the elements of $\mathcal{T}$.

That their union exhausts E follows from the assumption that every $\mathcal{T}_x$ is bounded, hence included in some $B_c(0_F, n)$.

(c) It follows from Baire's Theorem that at least one of the sets E_n has non-empty interior, hence contains an open ball. Denoting by n_0 the index of that particular non-meager set, x_0 the center of the ball and r its radius immediately gives the desired statement.

(d) The previous result shows that the operators $T \in \mathcal{T}$ are uniformly bounded on the open ball $B(x_0, r)$ hence also on the closed ball $B_c(x_0, \rho)$ with $0 < \rho < r$.

By linearity of the elements of $\mathcal{T}$ and isometry of translations, they are also uniformly bounded on $B_c(0_E, \rho)$ by some constant M, from which it follows that $\dfrac{M}{\rho}$ is a joint Lipschitz constant for all $T \in \mathcal{T}$.