Linear Operators on Banach Spaces

Due Apr. 8

Problem 1. Consider, for $x_0 \in [a,b]$, the linear map $\mathcal{I}_{x_0} : C([a,b]) \longrightarrow C([a,b])$ defined by

$$\mathcal{I}_{x_0} f(x) = \int_{x_0}^{x} f(t)\, dt.$$

(a) Prove that $\mathcal{I}_{x_0}$ is a bounded operator and that $\|\mathcal{I}_{x_0}\|_{\text{op}} \leq b - a$.

(b) Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of functions with continuous derivatives such that:

- the sequence $\{f_n'\}_{n\in\mathbf{N}}$ converges uniformly on $[a,b]$;
- there exists a point $x_0 \in [a,b]$ such that $\{f_n(x_0)\}_{n\in\mathbf{N}}$ converges.

Prove that $\{f_n\}_{n\in\mathbf{N}}$ converges in $C([a,b])$.

Problem 2. Denote by E the Banach space $C([0,1])$ equipped with supremum norm and let $E^1 = C^1([0,1])$ be the subspace of all functions that admit a continuous derivative.

The purpose of this problem is to study properties of the differentiation operator

$$\begin{array}{rcl} D : E^1 & \longrightarrow & E \\ f & \longmapsto & f' \end{array}.$$

(a) Is D bounded?

(b) Is D closed?

(c) Is E^1 closed in E? If not, determine its closure.

Let F be a closed subspace of E, contained in E^1.

(d) Prove that the restriction of D to F is Lipschitz.

(e) Prove that F is necessarily finite-dimensional.

Problem 3. Let E be a Banach space, F a normed linear space and $\mathcal{T}$ a family of bounded operators between E and F. The purpose of this problem is to prove the equivalence of the following statements:

(i) $\mathcal{T}$ is bounded in $\mathcal{B}(E,F)$.

(ii) For every x in E, the family $\mathcal{T}_x = \{Tx \,:\, T \in \mathcal{T}\}$ is bounded in F.

(a) Check that (i) implies (ii).

For $n \in \mathbf{N}$, let

$$E_n = \{x \in E \,:\, \|Tx\| \leq n \quad \text{for all } T \text{ in } \mathcal{T}\}.$$

(b) Verify that the sets E_n are closed and that $E = \bigcup_{n\in\mathbf{N}} E_n$.

(c) Prove the existence of a point $x_0 \in E$, a real number $r > 0$ and an integer n_0 such that

$$\forall T \in \mathcal{T}\,,\ \|x - x_0\| < r \quad \Rightarrow \quad \|Tx\| \leq n_0.$$

(d) Find a common Lipschitz constant for all T in $\mathcal{T}$ and conclude.