Normed Linear Spaces

Elements of Solution

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1. (*Classic norms comparison.*) Recall that two norms N_1 and N_2 on a linear space E are said *equivalent* if there exist constants α and β such that

$$\alpha N_1(x) \leq N_2(x) \leq \beta N_1(x)$$

for all $x \in E$.

(a) Prove that the norms $\|\_\|_1$, $\|\_\|_2$ and $\|\_\|_\infty$ on $\mathbf{R}^n$ are equivalent.

(b) For $X = (x_1, \ldots, x_n)$ fixed in $\mathbf{R}^n$, determine $\lim_{p\to\infty} \|X\|_p$.

(c) Prove that equivalent norms have the same bounded sets.

Solution. **(a)** Let $X = (x_1, \ldots, x_n) \in \mathbf{R}^n$. It is clear that

$$\|X\|_\infty \leq \|X\|_1 \leq n\|X\|_\infty \qquad \text{and} \qquad \|X\|_\infty \leq \|X\|_2 \leq \sqrt{n}\|X\|_\infty,$$

showing that both $\|\_\|_1$ and $\|\_\|_2$ are equivalent to $\|\_\|_\infty$. The equivalence of $\|\_\|_1$ and $\|\_\|_2$ follows by transitivity, or from the observation that

$$\|X\|_2 \leq \|X\|_1 \leq \sqrt{n}\|X\|_2,$$

where the second inequality follows from Cauchy-Schwarz.

(b) Assume $X \neq 0$ and let $i_0 \in \{1, \ldots, n\}$ be such that $|x_{i_0}| = \max\{|x_i|, 1 \leq i \leq n\}$. Then,

$$\frac{\|X\|_p^p}{\|X\|_\infty^p} = \frac{|x_1|^p}{|x_{i_0}|^p} + \cdots + \frac{|x_n|^p}{|x_{i_0}|^p} \underset{p\to\infty}{\longrightarrow} K$$

where K is the number of indices $i \in \{1, \ldots, n\}$ such that $|x_i| = |x_{i_0}|$.

It follows that $p \ln\left(\frac{\|X\|_p}{\|X\|_\infty}\right)$ has a finite limit as $p \to \infty$, which implies that

$$\lim_{p\to\infty} \frac{\|X\|_p}{\|X\|_\infty} = 1,$$

that is, $\|X\|_p \underset{p\to\infty}{\longrightarrow} \|X\|_\infty$, thus justifying the standard notation.

(c) By definition, a subset of E is bounded if and only if it is contained in a ball, which may be assumed to be centered at the origin without loss of generality. Now if the relation

$$\alpha N_1(x) \leq N_2(x) \leq \beta N_1(x)$$

holds for all $x \in E$, then for any $r > 0$,

$$B_{N_1}(0_E, r) \subset B_{N_2}(0_E, \beta r) \qquad \text{and} \qquad B_{N_2}(0_E, r) \subset B_{N_1}\left(0_E, \frac{r}{\alpha}\right)$$

so that every set that is bounded for N_1 is bounded for N_2 and vice versa.

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2. (*Weighted norms*) Denote by $\mathcal{W}$ the set of non-negative continuous functions on $[0,1]$ that vanish finitely many times (possibly zero). For $\varphi \in \mathcal{W}$ and $f \in C\big([0,1]\big)$, let

$$\|f\|_\varphi = \sup_{x\in[0,1]} |f(x)\varphi(x)|.$$

(a) Prove that $\|\_\|_\varphi$ is a norm on $C\big([0,1]\big)$.

(b) Assume φ_1 and φ_2 are strictly positive elements of $\mathcal{W}$.

Prove that $\|\_\|_{\varphi_1}$ and $\|\_\|_{\varphi_2}$ are equivalent.

From now on, let $\varphi_1(x) = x$ and $\varphi_2(x) = x^2$.

(c) Verify that $\|f\|_{\varphi_2} \leq \|f\|_{\varphi_1}$ for every $f \in C\big([0,1]\big)$.

(d) Prove that $\|\_\|_{\varphi_1}$ and $\|\_\|_{\varphi_2}$ are not equivalent.

Hint: consider the sequence of functions $f_n : x \longmapsto (1-x)^n$.

Solution. **(a)** Notice that the pointwise product of f and φ is a continuous function on the compact set $[0,1]$ and is therefore bounded. It follows that

$$\|f\|_\varphi = \|f\varphi\|_\infty$$

from which straightforward calculations show that $\|\_\|_\varphi$ is a semi-norm. In addition, if $\|f\varphi\|_\infty = 0$, then $f\varphi$ must be constantly 0, hence $f(x) \neq 0$ for finitely many values of x. However, if $f(x_0) \neq 0$, the continuity of f implies that $f(x) \neq 0$ for every x in a neighborhood of x_0, a necessarily infinite set. Therefore $\|f\|_\varphi = 0$ only if $f \equiv 0$ and $\|\_\|_\varphi$ is a norm.

(b) If $\varphi_2(x) \neq 0$ for all x, then the function $\dfrac{\varphi_1}{\varphi_2}$ is continuous on $[0,1]$. Letting $M = \left\|\dfrac{\varphi_1}{\varphi_2}\right\|_\infty$, we see that $|f(x)\varphi_1(x)| \leq M|f(x)\varphi_2(x)|$ for every x in $[0,1]$ so that

$$\|f\|_{\varphi_1} \leq M\|f\|_{\varphi_2}.$$

Exchange φ_1 and φ_2 in the argument to get the other estimate and conclude to the equivalence of the norms.

(c) Notice that $x^2 \leq x$ for every x in $[0,1]$ and apply a similar argument as in the previous question.

(d) A direct calculation shows that the function $x(1-x)^n$ reaches its maximum at $\dfrac{1}{n+1}$ so that

$$\|f_n\|_{\varphi_1} = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n \sim \frac{1}{en}.$$

Similarly, the maximum of $x^2(1-x)^n$ on $[0,1]$ is attained at $\dfrac{2}{n+2}$ hence

$$\|f_n\|_{\varphi_2} = \left(\frac{2}{n+2}\right)^2\left(1 - \frac{2}{n+2}\right)^n \sim \frac{4}{e^2n^2}.$$

It follows that the sequence $\{n^2 f_n\}_{n\in\mathbf{N}}$ is bounded for $\|\_\|_{\varphi_2}$ but not for $\|\_\|_{\varphi_1}$, showing that the norms are not equivalent.

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3. (*Weighted norms, 2.*) If $A = \{a_n\}_{n\in\mathbf{N}}$ is a sequence of real numbers and $U = \{u_n\}_{n\in\mathbf{N}} \in \mathcal{B}(\mathbf{N},\mathbf{R})$ is a bounded sequence, denote by $N_A(U)$ the (possibly divergent) series

$$N_A(U) = \sum_{n\in\mathbf{N}} a_n|u_n|.$$

(a) Find a necessary and sufficient condition on A for N_A to be a norm on $\mathcal{B}(\mathbf{N},\mathbf{R})$.

(b) Under the condition of **(a)**, compare N_A and $\|\_\|_\infty$. Are they equivalent?

Solution. **(a)** Notice that $N_A(\delta_n) = a_n$ so that a_n must be strictly positive for all $n \in \mathbf{N}$. Evaluating N_A on a constant sequence also shows that the series $\sum_n a_n$ must be convergent. Conversely, N_A is a norm whenever $A \in \ell^1(\mathbf{N},\mathbf{R}_+^\times)$.

(b) Observe that $N_A(U) \leq \|A\|_1\|U\|_\infty$. However, the norms are not equivalent as can be seen by considering the family $\left\{\dfrac{1}{a_n}\delta_n\right\}_{n\in\mathbf{N}}$, bounded with respect to N_A (by 1) but not with respect to $\|\_\|_\infty$.

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4. (*A bounded linear map with an unbounded inverse.*) Denote by T the linear map from $\mathbf{R}^\mathbf{N}$ to itself defined by

$$T(u)_n = \frac{u_n}{n+1} \qquad , \qquad \text{for } u = \{u_n\}_{n\in\mathbf{N}}.$$

(a) Prove that T restricts to a bounded operator T_0 from $\big(c_{00}(\mathbf{N}), \|\_\|_\infty\big)$ to itself.

(b) Verify that T_0 is invertible. Is T_0^{-1} bounded?

(c) Discuss the restriction of T to $c_0(\mathbf{N})$.

(d) (Optional.) Identify $c_{00}(\mathbf{N})$ with the space of polynomial functions on $\mathbf{R}$. Express T_0 and T_0^{-1} under this identification.

<u>Hint</u>: denote by δ_n the sequence defined by $\delta_n(k) = 0$ for $k \neq n$ and $\delta_n(n) = 1$. Notice that it is a Hamel basis for $c_{00}(\mathbf{N})$.

Solution. **(a)** Notice that $T(u)$ and u have the same support so T maps $c_{00}(\mathbf{N})$ to itself. It is linear and 1-Lipschitz, hence a bounded operator.

(b) Intuitively, T_0 correspond in the canonical basis $\{\delta_n\}_{n\in\mathbf{N}}$ to the infinite diagonal matrix $\begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ & & & \ddots \end{bmatrix}$ hence it is invertible with inverse corresponding to the infinite diagonal matrix $\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \end{bmatrix}$ which is unbounded as it sends (unit) vectors of the canonical basis to arbitrarily large vectors. Concretely, one checks that T_0^{-1} is defined by

$$T_0^{-1}(u)_n = (n+1)u_n$$

which is unbounded because $\|T_0^{-1}(\delta_n)\|_\infty = n+1$ whereas $\|\delta_n\|_\infty = 1$.

(c) The restriction of T maps $c_0(\mathbf{N})$ to itself and it is still a bounded operator but it is not surjective: consider for instance the sequence $u_n = \sqrt{n}$.

(d) The linear map ι defined by $\iota : \delta_n \longmapsto X^n$ is an isomorphism and if P is a polynomial function on $\mathbf{R}$, then

$$\iota \circ T_0 \circ \iota^{-1}(P)(x) = \frac{1}{X}\int P \qquad \text{and} \qquad \iota \circ T_0^{-1} \circ \iota^{-1}(P) = (XP)'.$$