Normed Linear Spaces

Due Mar. 27

Problem 1. Recall that two norms N_1 and N_2 on a linear space E are said *equivalent* if there exist constants α and β such that

$$\alpha N_1(x) \leq N_2(x) \leq \beta N_1(x)$$

for all $x \in E$.

(a) Prove that the norms $\|\_\|_1$, $\|\_\|_2$ and $\|\_\|_\infty$ on $\mathbf{R}^n$ are equivalent.

(b) For $X = (x_1, \ldots, x_n)$ fixed in $\mathbf{R}^n$, determine $\lim\limits_{p \to \infty} \|X\|_p$.

(c) Prove that equivalent norms have the same bounded sets.

Problem 2. Denote by $\mathcal{W}$ the set of non-negative continuous functions on $[0,1]$ that vanish finitely many times (possibly zero). For $\varphi \in \mathcal{W}$ and $f \in C\big([0,1]\big)$, let

$$\|f\|_\varphi = \sup_{x \in [0,1]} |f(x)\varphi(x)|.$$

(a) Prove that $\|\_\|_\varphi$ is a norm on $C\big([0,1]\big)$.

(b) Assume φ_1 and φ_2 are strictly positive elements of $\mathcal{W}$.

Prove that $\|\_\|_{\varphi_1}$ and $\|\_\|_{\varphi_2}$ are equivalent.

From now on, let $\varphi_1(x) = x$ and $\varphi_2(x) = x^2$.

(c) Verify that $\|f\|_{\varphi_2} \leq \|f\|_{\varphi_1}$ for every $f \in C\big([0,1]\big)$.

(d) Prove that $\|\_\|_{\varphi_1}$ and $\|\_\|_{\varphi_2}$ are not equivalent.

<u>Hint</u>: consider the sequence of functions $f_n : x \longmapsto (1-x)^n$.

Problem 3. If $A = \{a_n\}_{n\in\mathbf{N}}$ is a sequence of real numbers and $U = \{u_n\}_{n\in\mathbf{N}} \in \mathcal{B}(\mathbf{N},\mathbf{R})$ is a bounded sequence, denote by $N_A(U)$ the (possibly divergent) series

$$N_A(U) = \sum_{n\in\mathbf{N}} a_n|u_n|.$$

(a) Find a necessary and sufficient condition on A for N_A to be a norm on $\mathcal{B}(\mathbf{N},\mathbf{R})$.

(b) Under the condition of **(a)**, compare N_A and $\|\_\|_\infty$. Are they equivalent?

Problem 4. Denote by T the linear map from $\mathbf{R}^{\mathbf{N}}$ to itself defined by

$$T(u)_n = \frac{u_n}{n+1} \qquad , \qquad \text{for } u = \{u_n\}_{n\in\mathbf{N}}.$$

(a) Prove that T restricts to a bounded operator T_0 from $\big(c_{00}(\mathbf{N}), \|\_\|_\infty\big)$ to itself.

(b) Verify that T_0 is invertible. Is T_0^{-1} bounded?

(c) Discuss the restriction of T to $c_0(\mathbf{N})$.

(d) (Optional.) Identify $c_{00}(\mathbf{N})$ with the space of polynomial functions on $\mathbf{R}$. Express T_0 and T_0^{-1} under this identification.

Hint: denote by δ_n the sequence defined by $\delta_n(k) = 0$ for $k \neq n$ and $\delta_n(n) = 1$. Notice that it is a Hamel basis for $c_{00}(\mathbf{N})$.