Applications of Baire's Theorem

Elements of Solution

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1. (*Baire's Simple Limit Theorem*) Let (E, d) and (F, δ) be metric spaces.

Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

(a) Consider, for $n \geq 1$ and $\varepsilon > 0$, the set $F_{n,\varepsilon} = \{x \in E\ ,\ \forall p \geq n\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$.

Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

(b) Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N}\ ,\ \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

(c) Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

Solution. **(a)** According to the corollary of Baire's Theorem proved in class, it suffices to prove that the sets $F_{n,\varepsilon}$ are closed and cover E. For given n and p, the set $\{x \in E\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$ is closed as the inverse image of $[0, \varepsilon]$, closed, under the map $x \mapsto \delta(f_n(x), f_p(x))$, continuous as composed of continuous functions. Taking the intersection over $p \geq n$ gives $F_{n,\varepsilon}$ closed. That the union of these sets covers E follows from the pointwise convergence of the sequence $\{f_n\}_{n\in\mathbf{N}}$.

(b) Let n be such that $x_0 \in \overset{\circ}{F}_{n,\varepsilon}$. Since $\overset{\circ}{F}_{n,\varepsilon}$ is open and f_n is continuous, there exists a neighborhood $\mathcal{N}$ of x_0 included in $\overset{\circ}{F}_{n,\varepsilon}$ such that

$$\delta(f_n(x_0), f_n(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Since $\mathcal{N} \subset \overset{\circ}{F}_{n,\varepsilon}$, we have

$$\delta(f_n(x), f_p(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N} \text{ and } p \geq n.$$

Letting $p \to \infty$ in this inequality, we get

$$\delta(f_n(x), f(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Now, by the triangle inequality,

$$\begin{aligned}\delta(f(x), f(x_0)) &\leq \delta(f(x), f_n(x)) + \delta(f_n(x), f_n(x_0)) + \delta(f_n(x_0), f(x_0)) \\ &\leq \varepsilon + \varepsilon + \varepsilon\end{aligned}$$

for all $x \in \mathcal{N}$.

(c) Let $x_0 \in \Omega$ and $\varepsilon > 0$. Fix n such that $\frac{1}{n} < \frac{\varepsilon}{3}$. By the previous result, there is a neighborhood $\mathcal{N}$ of x_0 such that $\delta(f(x), f(x_0)) \leq \varepsilon$ for all $x \in \mathcal{N}$, which proves continuity of f at x_0. The fact that Ω is dense in E follows from **(a)** and the Baire Category Theorem.

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2. (*Continuity of derivatives.*) Let f be differentiable on $\mathbf{R}$. Show that f' is continuous on a dense set.

Hint: Apply the result of the previous problem to a well-chosen sequence.

Solution. Apply the previous result to the sequence $(f_n)_{n\geq 1}$ defined by:

$$f_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{1/n}.$$

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3. (*Hermite's Dreadful Plague.*) The purpose of this problem is to show that nowhere differentiable functions are dense in $E = C\big([0,1], \mathbf{R}\big)$ equipped with its ordinary norm.

Consider, for $\varepsilon > 0$ and $n \in \mathbf{N}$,

$$U_{n,\varepsilon} = \left\{ f \in E \,,\, \forall x \in [0,1] \,,\, \exists y \in [0,1] \,,\, |x - y| < \varepsilon \quad \text{and} \quad \left| \frac{f(y) - f(x)}{y - x} \right| > n \right\}.$$

(a) Prove that every set $U_{n,\varepsilon}$ has a close complement.

For $p \geq 1$ integer, let v_p be a continuous function on $[0,1]$, affine on each interval $\left[\dfrac{k}{2p}, \dfrac{k+1}{2p}\right]$ and such that $v_p\left(\frac{k}{2p}\right) = 0$ (resp. $= 1$) if k is even (resp. odd).

(b) Sketch the graphs of v_1, v_2 and v_3.

Let f be a function of class C^1 on $[0,1]$ and $g_p = f + \lambda v_p$ with $\lambda > 0$.

(c) Verify that g_p can be chosen arbitrarily close to f in $C\big([0,1], \mathbf{R}\big)$.

(d) Prove that

$$\left| \frac{g_p(x) - g_p(y)}{x - y} \right| \geq \lambda \left| \frac{v_p(x) - v_p(y)}{x - y} \right| - \|f'\|_\infty$$

for $x \neq y$ in $[0,1]$

(e) Verify that $g_p \in U_{n,\varepsilon}$ whenever $p > \dfrac{n + \|f'\|_\infty}{2\lambda}$.

(f) Prove that $U_{n,\varepsilon}$ is dense in E.

(g) Conclude.

Solution. **(a)** Observe that

$$U^c_{n,\varepsilon} = \left\{ f \in E \,,\, \exists x \in [0,1] \,,\, \forall y \in [0,1] \,,\, |x-y| < \varepsilon \Rightarrow \left| \frac{f(y)-f(x)}{y-x} \right| \le n \right\}.$$

and let $\{f_k\}$ be a sequence in $U^c_{n,\varepsilon}$ that converges to f in E. For each k, there exists $x_k \in [0,1]$ such that $|x_k - y| < \varepsilon \Rightarrow \left|\frac{f(y)-f(x_k)}{y-x_k}\right| \le n$. Since $[0,1]$ is compact, $\{x_k\}$ has a convergent subsequence $\{x_{\varphi(k)}\}$. Denote x its limit and let y in $[0,1]$ be such that $0 < |x-y| < \varepsilon$. For k large enough, one has $0 < |x_{\varphi(k)} - y| < \varepsilon$ so that $\left|\frac{f_{\varphi(k)}(y) - f_{\varphi(k)}(x_{\varphi(k)})}{y - x_{\varphi(k)}}\right| \le n$ and the uniform convergence $f_{\varphi(k)} \to f$ implies that $\left| \dfrac{f(y)-f(x)}{y-x} \right| \le n$, so that f belongs to $U^c_{n,\varepsilon}$.

(c) The functions g_p are continuous as combinations of continuous functions and $\|f - g_p\|_\infty \le \lambda$.

(d) If $x \neq y$ in $[0,1]$, then

$$\begin{aligned} \left| \frac{g_p(x) - g_p(y)}{x-y} \right| &\ge \lambda \left| \frac{v_p(x) - v_p(y)}{x-y} \right| - \left| \frac{f(x)-f(y)}{x-y} \right| \\ &\ge \lambda \left| \frac{v_p(x) - v_p(y)}{x-y} \right| - \|f'\|_\infty. \end{aligned}$$

(e) Let $p > \dfrac{n + \|f'\|_\infty}{2\lambda}$. For any $x \in [0,1]$, there exists $y \in [0,1]$ within ε of x and in the same interval $\left[\frac{k}{2p}, \frac{k+1}{2p}\right]$. By definition of v_p, the latter implies that $\left|\frac{v_p(x)-v_p(y)}{x-y}\right| = 2p$. Then

$$\left| \frac{g_p(x) - g_p(y)}{x-y} \right| \ge 2p\lambda - \|f'\|_\infty > n$$

so that $g_p \in U_{n,\varepsilon}$.

(f) Polynomials are dense in E, and we just proved that functions of class C^1 can be approximated by elements of $U_{n,\varepsilon}$, which is therefore dense in E.

(g) The Baire Category Theorem ensures that $U = \bigcap_{n \ge 1} U_{\frac{1}{n},n}$ is dense in E. Let $f \in U$ and $x \in [0,1]$. Then there is a sequence $\{x_n\}$ such that

$$0 < |x_n - x| < \frac{1}{n} \quad \text{and} \quad \left| \frac{f(x_n) - f(y)}{x_n - y} \right| > n,$$

which prevents f from being differentiable at x.

'Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui n'ont point de dérivées.'

C. Hermite, 1893.