Applications of Baire's Theorem

Due Mar. 3rd

Problem 1. Let (E, d) and (F, δ) be metric spaces.

Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

(a) Consider, for $n \geq 1$ and $\varepsilon > 0$, the set $F_{n,\varepsilon} = \{x \in E \, , \, \forall p \geq n \, , \, \delta(f_n(x), f_p(x)) \leq \varepsilon\}$.

Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

(b) Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N} \, , \, \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

(c) Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

Problem 2. Let f be differentiable on $\mathbf{R}$. Show that f' is continuous on a dense set.

Hint: Apply the result of the previous problem to a well-chosen sequence.

Problem 3. (*Optional.*) The purpose of this problem is to show that nowhere differentiable functions are dense in $E = C\big([0,1], \mathbf{R}\big)$ equipped with its ordinary norm.

Consider, for $\varepsilon > 0$ and $n \in \mathbf{N}$,

$$U_{n,\varepsilon} = \left\{ f \in E \, , \, \forall x \in [0,1] \, , \, \exists y \in [0,1] \, , \, |x - y| < \varepsilon \quad \text{and} \quad \left| \frac{f(y) - f(x)}{y - x} \right| > n \right\}.$$

(a) Prove that every set $U_{n,\varepsilon}$ has a closed complement.

For $p \geq 1$ integer, let v_p be a continuous function on $[0,1]$, affine on each interval $\left[\frac{k}{2p}, \frac{k+1}{2p}\right]$ and such that $v_p\left(\frac{k}{2p}\right) = 0$ (resp. $= 1$) if k is even (resp. odd).

(b) Sketch the graphs of v_1, v_2 and v_3.

Let f be a function of class C^1 on $[0,1]$ and $g_p = f + \lambda v_p$ with $\lambda > 0$.

(c) Verify that g_p can be chosen arbitrarily close to f in $C\big([0,1], \mathbf{R}\big)$.

(d) Prove that

$$\left|\frac{g_p(x) - g_p(y)}{x - y}\right| \geq \lambda \left|\frac{v_p(x) - v_p(y)}{x - y}\right| - \|f'\|_\infty$$

for $x \neq y$ in $[0,1]$

(e) Verify that $g_p \in U_{n,\varepsilon}$ whenever $p > \dfrac{n + \|f'\|_\infty}{2\lambda}$.

(f) Prove that $U_{n,\varepsilon}$ is dense in E.

(g) Conclude.