Completeness - Topology in $C(X)$

Elements of Solution

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1. (*Summable sequences.*) Consider the space $\ell^1(\mathbf{N})$ of sequences $U = \{u_n\}_{n\in\mathbf{N}}$ such that the series $\sum_{n\geq 0} |u_n|$ converges, equipped with the distance d_1 associated with the norm

$$\|U\|_1 = \sum_{n\geq 0} |u_n|.$$

(a) Prove that $\ell^1(\mathbf{N})$ is complete.

Consider the subspace $c_{00}(\mathbf{N})$ of finitely supported sequences.

(b) Is it complete for d_1? If not, determine its completion.

Solution. **(a)** Let $\{U^k\}_{k\in\mathbf{N}}$ be a Cauchy sequence in $\ell^1(\mathbf{N})$. Let $\varepsilon > 0$ and consider p and q large enough so that $d_1(U^p - U^q) < \varepsilon$, that is,

$$(\star) \qquad \sum_{n\in\mathbf{N}} |u_n^p - u_n^q| < \varepsilon.$$

In particular, $|u_n^p - u_n^q| < \varepsilon$ for any $p, q, n \in \mathbf{N}$, showing that, for any fixed n, the sequence $\{u_n^k\}_{k\in\mathbf{N}}$ is Cauchy in $\mathbf{R}$ complete. Denote by $U = \{u_n\}_{n\in\mathbf{N}}$ the pointwise limit defined by

$$u_n = \lim_{k\to\infty} u_n^k.$$

Let us prove that $U \in \ell^1(\mathbf{N})$. Cauchy sequences are bounded, so there exists $M \geq 0$ such that $\|U^k\|_1 \leq M$ for all k. In particular, partial sums are bounded by M as well: for any $N \geq 1$,

$$\sum_{n=0}^{N} |u_n^k| \leq \|U^k\|_1 \leq M.$$

Letting $k \to \infty$ in each of these finite sums, we see that $\sum_{n=0}^{N} |u_n| \leq M$ for any N, so that the series associated with U converges absolutely and $\|U\|_1 \leq M$.

Finally, we prove that $d_1(U^k, U) \to 0$ as $k \to \infty$. Fix $\varepsilon > 0$ and consider p and q large enough for $(\star)$ to hold. Fixing p and letting $q \to \infty$ gives $d_1(U^p, U) \leq \varepsilon$, and since ε

can be chosen arbitrarily small, it shows that $\{U^k\}_{k\in\mathbf{N}}$ converges in $\ell^1(\mathbf{N})$, which is therefore complete.

(b) The example of a convergent geometric series shows that $c_{00}(\mathbf{N})$ is a strict subset of $\ell^1(\mathbf{N})$. We will show that it is dense in $\ell^1(\mathbf{N})$, establishing at the same time that it is not closed, hence not complete and that its completion with respect to d_1 is $\ell^1(\mathbf{N})$.

Consider, for $U = \{u_n\}_{n\in\mathbf{N}}$ in $\ell^1(\mathbf{N})$, the sequence $\{U^k\}_{k\in\mathbf{N}}$ defined by truncation:

$$u_n^k = \begin{cases} u_n & \text{if } n \leq k \\ 0 & \text{otherwise} \end{cases}.$$

Then, by construction

$$d_1(U^k, U) = \sum_{n\geq k+1} |u_n|,$$

which converges to 0 as $k \to \infty$ because U is summable, showing that U is the limit in $\ell^1(\mathbf{N})$ of the sequence $\{U^k\}_{k\in\mathbf{N}}$ of elements of $c_{00}(\mathbf{N})$.

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2. (*Completeness is not a topological property.*) Let $E = (0, +\infty)$ and for $x, y \in E$, consider $\delta(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

(a) Prove that δ is a distance on E and that it induces the same topology as the Euclidean distance d.

(b) Is the map $x \mapsto x^{-1}$ uniformly continuous as a map from (E, d) to itself ? As a map from (E, d) to (E, δ)?

(c) Is (E, δ) complete ?

(d) Is $((0, 1], d)$ complete?

(e) Is $((0, 1], \delta)$ complete?

(f) Is completeness a topological property?

Solution. **(a)** *Method 1*: prove that every open d-ball contains a δ-ball with the same center and vice versa. *Method 2*: prove that $(E, d) \xrightarrow{\text{Id}} (E, \delta)$ is a homeomorphism. To see this, it is convenient to decompose the identity map as $(E, d) \xrightarrow{\varphi} (E, d) \xrightarrow{\varphi} (E, \delta)$ where $\varphi(x) = x^{-1}$ and prove that both are homeomorphisms. Note that both methods boil down to the fact that φ is a homeomorphism from $(E, d \,\text{or}\, \delta)$ to $(E, d \,\text{or}\, \delta)$.

(b) No. Yes.

(c) No: $u_n = n$ is Cauchy but it does not converge (argue by contradiction).

(d) No: it is not closed in $(\mathbf{R}, d)$. Alternatively, consider $u_n = \frac{1}{n}$, Cauchy but not convergent in $(0, 1]$.

(e) Yes. *Method 1*: show that a Cauchy sequence $\{u_n\}$ for δ is also Cauchy for d hence converges for d in the closure of $(0, 1]$. If the d-limit is > 0, it is also the δ-limit because d and δ induce the same topology (or check it directly with balls) so the sequence converges. Assume the limit is 0. Then $\delta(1, u_n)$ diverges to $+\infty$ so $\{u_n\}$ is not bounded which is impossible since it is Cauchy. In conclusion, a Cauchy sequence in $((0, 1], \delta)$ converges in $((0, 1], \delta)$, which is therefore complete.

Method 2: $x \mapsto x^{-1}$ is an isometric (hence uniformly continuous) homeomorphism between $((0, 1], \delta)$ and $([1, +\infty), d)$, which is closed in $(\mathbf{R}, d)$ complete, hence complete. Uniformly continuous homeomorphisms preserve completeness so $((0, 1], \delta)$ is complete.

(f) No: the results of **(a)**, **(d)** and **(e)** prove that $((0, 1], d)$ and $((0, 1], \delta)$ are homeomorphic (through Id), but only the latter is complete.

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3. (*Weierstrass' Theorem via Dirac sequences.*) If f and g are functions on the real line, their *convolution* is the function $f \star g$ defined by

$$\big(f \star g\big)(x) = \int_{\mathbf{R}} f(x - t) g(t)\, dt$$

if it makes sense.

(a) Verify that $\star$ is well-defined on the space $C_c(\mathbf{R})$ of continuous functions on $\mathbf{R}$ that vanish outside of a compact subset, and that it is commutative and distributive with respect to addition.

A *Dirac sequence* is a sequence $\{\varphi_n\}_{n \in \mathbf{N}}$ of non-negative functions in $C_c(\mathbf{R})$, satisfying the following conditions

$$(1) \qquad \forall n \in \mathbf{N}\,, \int_{\mathbf{R}} \varphi_n(t)\, dt = 1$$

$$(2) \qquad \forall \alpha > 0\,, \lim_{n \to \infty} \int_{\mathbf{R} \setminus [-\alpha, \alpha]} \varphi_n(t)\, dt = 0$$

(b) Assume given a Dirac sequence $\{\varphi_n\}_{n \in \mathbf{N}}$. Prove that if $f \in C_c(\mathbf{R})$, then $\varphi_n \star f$ converges uniformly to f.

<u>Hint</u>: Continuous functions with compact support are uniformly continuous.

For $n \in \mathbf{N}$, consider

$$a_n = \int_{-1}^{1} (1 - t^2)^n\, dt \qquad \text{and} \qquad \pi_n : t \longmapsto \begin{cases} (1 - t^2)^n / a_n & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Prove that $\{\pi_n\}_{n\in\mathbf{N}}$ is a Dirac sequence.

(d) Assume that $f \in C_c(\mathbf{R})$ is supported in the segment $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Prove that $f \star \pi_n$ is a polynomial function on I.

(e) Derive Weierstrass' Approximation Theorem: every continuous function on the segment $[a, b]$ is the uniform limit of a sequence of polynomial functions.

Solution. **(a)** The integrand defining $f \star g$ is compactly supported if f and g are so the convolution is well-defined. Distributivity follows from the distributivity of multiplication over addition in $\mathbf{R}$ and the additivity of integrals. Use the change of variables $u = x - t$ to verify that $\star$ is commutative.

(b) Let $\varepsilon > 0$. Since f is continuous and compactly supported, it is uniformly continuous. Therefore, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

For $x \in \mathbf{R}$,

$$\begin{aligned} |f \star \varphi_n(x) - f(x)| &= \left|\int_{\mathbf{R}} f(x-t)\varphi_n(t)\,dt - \int_{\mathbf{R}} f(x)\varphi_n(t)\,dt\right| \\ &= \left|\int_{\mathbf{R}} \big(f(x-t) - f(x)\big)\varphi_n(t)\,dt\right| \\ &\leq \int_{-\delta}^{\delta} |f(x-t) - f(x)|\,\varphi_n(t)\,dt + \int_{\mathbf{R}\setminus(-\delta,\delta)} |f(x-t) - f(x)|\,\varphi_n(t)\,dt. \end{aligned}$$

Condition (2) implies that $\displaystyle\int_{\mathbf{R}\setminus[-\alpha,\alpha]} \varphi_n(t)\,dt < \varepsilon$ when n is large enough, in which case

$$|f \star \varphi_n(x) - f(x)| \leq \varepsilon \int_{-\delta}^{\delta} \varphi_n(t)\,dt + 2\|f\|_\infty \varepsilon \leq \big(1 + 2\|f\|_\infty\big)\varepsilon.$$

The upper bound is independent of x and can be made arbitrarily small for n large enough, so $\|f \star \varphi_n - f\|_\infty \underset{n\to\infty}{\longrightarrow} 0$.

(c) Non-negativity is immediate and the normalization condition (1) follows from a direct calculation. Note that $\displaystyle\int_{\mathbf{R}\setminus[-\alpha,\alpha]} \pi_n(t)\,dt = 0$ if $\alpha \geq 1$ and that

$$a_n = 2\int_0^1 (1 - t^2)^n\,dt \geq 2\int_0^1 (1-t)^n\,dt = \frac{2}{n+1}.$$

For $0 < \alpha < 1$ and $n \geq 1$, we see that

$$\begin{aligned}\int_{\mathbf{R}\setminus[-\alpha,\alpha]} \pi_n(t)\,dt &= \frac{2}{a_n}\int_\alpha^1 (1-t^2)^n\,dt \\ &\leq \frac{2}{a_n}(1-\alpha^2)^n \leq (n+1)(\underbrace{1-\alpha^2}_{<1})^n \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

(d) First observe that $\pi_n(x-t)$ is polynomial in x. To fix notations, we write

$$\pi_n(x-t) = \sum_{k=0}^{2n} c_k(t)x^k.$$

Then, for x in the support of the convolution,

$$(f*\pi_n)(x) = \sum_{k=0}^{2n}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)c_k(t)\,dt\right)x^k$$

which is a polynomial expression.

(e) It follows from the previous results that a continuous function with compact support in $\left[-\frac{1}{2},\frac{1}{2}\right]$ is a uniform limit of polynomial functions. Now let f be a continuous function defined on a segment $[a,b]$. Then f extends to a compactly supported continuous function $\tilde{f}$: consider for instance $\tilde{f}$ to be 0 oustide of $[a-1,b+1]$, equal to f on $[a,b]$ and affine elsewhere.

Now consider the affine transformation $h : t \longmapsto (b-a+2)t + \dfrac{a+b}{2}$. It is a homeomorphism between $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $[a-1,b+1]$ so the composition $f\circ h$ is continuous and compactly supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$. Therefore, there exists a sequence $\{p_n\}_{n\in\mathbf{N}}$ of polynomial functions that converges uniformly on $\left[-\frac{1}{2},\frac{1}{2}\right]$ to $f\circ h$.

Notice that h^{-1} is also an affine function so that $\{p_n\circ h^{-1}\}_{n\in\mathbf{N}}$ is a sequence of polynomial functions on $[a,b]$. We will prove that this sequence converges uniformly to f. For $x\in[a,b]$,

$$|p_n\circ h^{-1}(x) - f(x)| = |p_n\circ h^{-1}(x) - f\circ h\big(h^{-1}(x)\big)| \leq \|p_n - f\circ h\|_\infty.$$

The upper bound is independent of x and converges to 0 as $n\to\infty$, showing that $p_n\circ h^{-1} \underset{n\to\infty}{\longrightarrow} f$ uniformly on $\left[-\frac{1}{2},\frac{1}{2}\right]$.

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4. (*Uniform limits of polynomial functions.*) Let $\{P_n\}_{n\in\mathbf{N}}$ be a sequence of polynomial functions on $\mathbf{R}$.

(a) Prove that if P_n converges uniformly on $\mathbf{R}$ to a function f, then f is polynomial.

(b) Compare with the previous problem.

Solution. **(a)** If it converges, then $\{P_n\}_{n\in\mathbf{N}}$ must be a Cauchy sequence, so there exists an integer N such that $\|P_N - P_n\|_\infty < 1$ for any $n \geq N$. The polynomials $P_n - P_N$ are therefore bounded, hence constant. In other words there are constants α_n such that

$$P_n = P_N + \alpha_n$$

for $n \geq N$. The sequence $\{\alpha_n\}_{n\geq N}$ is Cauchy in $\mathbf{R}$ hence converges to some $\alpha \in \mathbf{R}$ and

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} P_N + \alpha_n = P_N + \alpha$$

is a polynomial function.

(b) Weierstrass' Theorem asserts that any continuous function on a compact interval of $\mathbf{R}$ is a uniform limit of polynomial functions, meaning that polynomial functions are dense. On the entire real line however, uniform limits of polynomial functions are polynomial, so polynomial functions are a closed subset.

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5. (*Hölder maps.*) A function $f \in C([0,1])$ is said to be α-*Hölder* if

$$h_\alpha(f) = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha,M} = \{f \in C([0,1],\mathbf{R})\,,\ h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M\}\,.$$

(a) Prove that $H_{\alpha,M}$ is compact in $(C([0,1],\mathbf{R}), \|\cdot\|_\infty)$.

(b) State Hölder's Inequality for functions on $\mathbf{R}_+$.

(c) Assume $f \in L^p(\mathbf{R}_+)$ with $p > 1$ and define F by:

$$F(x) = \int_0^x f(t)\,dt.$$

Prove that F is $\left(1 - \dfrac{1}{p}\right)$-Hölder.

Solution. **(a)** The Arzelà-Ascoli Theorem implies that it suffices to check that $H_{\alpha,M}$ is closed, bounded and equicontinuous. The set in question is the intersection of the closed ball $B_c(0,M)$ and $F_M = \{f \in C([0,1])\,,\ h_\alpha(f) \leq M\}$, so it is automatically bounded and it is enough to check that F_M is closed. To do so, consider a sequence

$\{f_n\}$ of functions in F_M, that converges to f in $C([0,1])$. The pointwise convergence of the sequence implies that

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq M$$

for every $x \neq y$ so F_M is closed.

To establish equicontinuity, let $\varepsilon > 0$ and verify that $\delta = \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}$ is an appropriate modulus of continuity.

(b) If $u \in L^p(\mathbf{R}_+)$ and $v \in L^q(\mathbf{R}_+)$ with p and q conjugate, that is, satisfying $\dfrac{1}{p}+\dfrac{1}{q} = 1$, the pointwise product uv is integrable on $\mathbf{R}^+$ and

$$\|uv\|_1 \leq \|u\|_p \, \|v\|_q.$$

(c) Let $0 \leq x < y$ and apply Hölder's Inequality with $u = f$ and $v = \mathbf{1}_{[x,y]}$.

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6. (*Weierstrass' Theorem.*) Look up and summarize a different proof of Weierstrass' Approximation Theorem.

Solution. A classical proof consists in considering Berstein polynomials: for f continuous on $[0,1]$, one can prove that the sequence of polynomials $\{B_n(f)\}_{n\geq 1}$ defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

converges to f uniformly on $[0,1]$.

A more general statement, due to M. Stone, characterizes dense subalgebras of $C_0(X)$ where X is a locally compact Hausdorff space.