Completeness - Topology in $C(X)$

Due Feb. 20

Problem 1. Consider the space $\ell^1(\mathbf{N})$ of sequences $U = \{u_n\}_{n\in\mathbf{N}}$ such that the series $\sum_{n\geq 0} |u_n|$ converges, equipped with the distance d_1 associated with the norm

$$\|U\|_1 = \sum_{n\geq 0} |u_n|.$$

(a) Prove that $\ell^1(\mathbf{N})$ is complete.

Consider the subspace $c_{00}(\mathbf{N})$ of finitely supported sequences.

(b) Is it complete for d_1? If not, determine its completion.

Problem 2. Let $E = (0, +\infty)$ and for $x, y \in E$, consider $\delta(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

(a) Prove that δ is a distance on E and that it induces the same topology as the Euclidean distance d.

(b) Is the map $x \mapsto x^{-1}$ uniformly continuous as a map from (E, d) to itself ? As a map from (E, d) to (E, δ)?

(c) Is (E, δ) complete ?

(d) Is $((0, 1], d)$ complete?

(e) Is $((0, 1], \delta)$ complete?

(f) Is completeness a topological property?

Problem 3. If f and g are functions on the real line, their *convolution* is the function $f \star g$ defined by

$$\big(f \star g\big)(x) = \int_{\mathbf{R}} f(x-t)g(t)\,dt$$

if it makes sense.

(a) Verify that $\star$ is well-defined on the space $C_c(\mathbf{R})$ of continuous functions on $\mathbf{R}$ that vanish outside of a compact subset, and that it is commutative and distributive with respect to addition.

A *Dirac sequence* is a sequence $\{\varphi_n\}_{n\in\mathbf{N}}$ of non-negative functions in $C_c(\mathbf{R})$, satisfying the following conditions

$$(1) \qquad \forall n \in \mathbf{N}\,,\ \int_{\mathbf{R}} \varphi_n(t)\,dt = 1$$

$$(2) \qquad \forall \alpha > 0\,,\ \lim_{n\to\infty} \int_{\mathbf{R}\setminus[-\alpha,\alpha]} \varphi_n(t)\,dt = 0$$

(b) Assume given a Dirac sequence $\{\varphi_n\}_{n\in\mathbf{N}}$. Prove that if $f \in C_c(\mathbf{R})$, then $\varphi_n \star f$ converges uniformly to f.

<u>Hint</u>: Continuous functions with compact support are uniformly continuous.

For $n \in \mathbf{N}$, consider

$$a_n = \int_{-1}^{1} (1-t^2)^n\,dt \qquad \text{and} \qquad \pi_n : t \longmapsto \begin{cases} (1-t^2)^n/a_n & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Prove that $\{\pi_n\}_{n\in\mathbf{N}}$ is a Dirac sequence.

(d) Assume that $f \in C_c(\mathbf{R})$ is supported in the segment $I = \left[-\frac{1}{2}\,,\,\frac{1}{2}\right]$. Prove that $f \star \pi_n$ is a polynomial function on I.

(e) Derive Weierstrass' Approximation Theorem: every continuous function on the segment $[a,b]$ is the uniform limit of a sequence of polynomial functions.

Problem 4. Let $\{P_n\}_{n\in\mathbf{N}}$ be a sequence of polynomial functions on $\mathbf{R}$.

(a) Prove that if P_n converges uniformly on $\mathbf{R}$ to a function f, then f is polynomial.

(b) Compare with the previous problem.

Problem 5. A function $f \in C([0,1])$ is said to be α-*Hölder* if

$$h_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha,M} = \{f \in C([0,1], \mathbf{R})\,,\ h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M\}\,.$$

(a) Prove that $H_{\alpha,M}$ is compact in $(C([0,1], \mathbf{R}), \|\cdot\|_\infty)$.

(b) State Hölder's Inequality for functions on $\mathbf{R}_+$.

(c) Assume $f \in L^p(\mathbf{R}_+)$ with $p > 1$ and define F by:

$$F(x) = \int_0^x f(t)\,dt.$$

Prove that F is $\left(1 - \dfrac{1}{p}\right)$-Hölder.

Problem 6. (*Optional.*) Look up and summarize a different proof of Weierstrass' Approximation Theorem.