Metric Topology

Elements of Solution

1. (*Product metric.*) Let (E_1, d_1) and (E_2, d_2) be metric spaces. Prove that the map defined by

$$d\big((x_1, x_2)\,,\,(y_1, y_2)\big) = \max\big(d_1(x_1, y_1)\,,\,d_2(x_2, y_2)\big)$$

is a metric on $E_1 \times E_2$.

Solution. Positivity, symmetry and separation are immediate. To check the triangle inequality, let (x_1, x_2), (y_1, y_2) and (z_1, z_2). The triangle inequalities for d_1 and d_2 give:

$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \quad \text{and} \quad d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2)$$

so that

$$\begin{aligned} d\big((x_1, x_2)\,,\,(z_1, z_2)\big) &= \max\big(d_1(x_1, z_1)\,,\,d_2(x_2, z_2)\big) \\ &\leq \max\big(d_1(x_1, y_1) + d_1(y_1, z_1)\,,\,d_2(x_2, y_2) + d_2(y_2, z_2)\big) \\ &\leq \max\big(d_1(x_1, y_1)\,,\,d_2(x_2, y_2)\big) + \max\big(d_1(y_1, z_1)\,,\,d_2(y_2, z_2)\big) \\ &= d\big((x_1, x_2)\,,\,(y_1, y_2)\big) + d\big((y_1, y_2)\,,\,(z_1, z_2)\big), \end{aligned}$$

showing that d satisfies the triangle inequality and is therefore a metric on $E_1 \times E_2$.

2. (*Compactness in metric spaces.*) Recall that a subset X of a topological space E is said *compact* if any open cover of X admits a finite subcover. In this problem, we assume that E is a metric space.

(a) What is a *totally bounded* set?

(b) What is a *sequentially compact* set?

(c) State all the implications between total boundedness, sequential compactness and compactness for a general metric space.

(d) State the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem for $\mathbf{R}^n$ equipped with its ordinary (Euclidean) metric.

(e) What parts of these results hold in greater generality?

Solution. **(a)** A metric space X is said *totally bounded* if for every $\varepsilon > 0$, the open cover

$$\{B(x,\varepsilon)\,,\, x \in X\}$$

admits a finite subcover. In other words, X is contained in a finite union of balls of any radius.

(b) A metric space X is said *sequentially compact* if every sequence in X has a convergent subsequence.

(c) In a metric space:

$$\text{sequential compactness} \quad \Leftrightarrow \quad \text{compactness} \quad \Rightarrow \quad \text{total boundedness.}$$

(d) Heine-Borel Theorem. Let X be a subset of $\mathbf{R}^n$ equipped with the metric associated with a norm. Then,

$$X \text{ is compact} \quad \Leftrightarrow \quad X \text{ is closed and bounded.}$$

Remarks about $(\Rightarrow)$:

- $($compact $\Rightarrow$ closed$)$ holds in every Hausdorff space, metric or not.
- $($compact $\Rightarrow$ bounded$)$ holds in every metric space.

A metric space is said to have the Heine-Borel property if its closed and bounded subsets are compact, that is, if $(\Leftarrow)$ holds.

- No infinite-dimensional Banach space has the Heine-Borel property.
- Some Fréchet spaces do: for instance $\mathrm{Hol}(\Omega)$ with Ω open in $\mathbf{C}^n$ or $C^\infty(\Omega)$ with Ω open in $\mathbf{R}^n$.

Bolzano-Weierstrass Theorem. Every bounded sequence in $\mathbf{R}^n$ equipped with the distance associated with a norm has a convergent subsequence. In other words:

$$X \text{ is sequentially compact} \quad \Leftrightarrow \quad X \text{ is closed and bounded.}$$

See Propositions 17, 18 and 19 in Section 9.5 of [Royden-Fitzpatrick] for proofs.

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3. (*Convexity of normed balls.*) Recall that a subset C of a linear space is said *convex* if for any x, y in C the line segment

$$[x,y] = \{(1-t)x + ty\,,\, 0 \le t \le 1\}$$

is included in C. Prove that balls in a normed linear space are always convex.

Solution. Let a be an element in the space and $r > 0$. If $x, y \in B(a,r)$ and $z_t = (1-t)x + ty$ with $t \in [0,1]$, then

$$\begin{aligned} d(a, z_t) &= \|a - (1-t)x - ty\| \\ &= \|(1-t)a + ta - (1-t)x - ty\| \\ &\leq \|(1-t)(a-x)\| + \|t(a-y)\| \\ &= (1-t)\|a-x\| + t\|a-y\| \\ &\leq (1-t)r + tr \\ &= r, \end{aligned}$$

showing that z_t belongs to $B(a,r)$ which is therefore convex.

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4. (*SNCF distance.*) Consider the map δ defined on $\mathbf{R}^2 \times \mathbf{R}^2$ by

$$\delta(u,v) = \begin{cases} \|u-v\| & \text{if } u \text{ and } v \text{ are colinear} \\ \|u\| + \|v\| & \text{otherwise} \end{cases}.$$

(a) Prove that δ is a distance.

(b) Describe geometrically the ball $B(u,r)$ for $u \in \mathbf{R}^2$ and $r > 0$.

(c) Is there a norm N on $\mathbf{R}^2$ such that $\delta(u,v) = N(u-v)$ for all $u, v \in \mathbf{R}^2$?

Solution. **(a)** For the triangle inequality, distinguish cases when two or three of the vectors are colinear.

(b) If $u = 0$, it is the Euclidean ball with radius r. Otherwise, it is the union of the open line segment $\left\{ u + t\dfrac{u}{\|u\|} \ , \ -1 < t < 1 \right\}$ and the (possibly empty) Euclidean ball $B(0,\rho)$ with $\rho = \max(r - \|u\|, 0)$.

(c) No: some of the balls for this distance are not convex, which cannot happen in a normed linear space, as established in a previous question.

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5. (*Metric Urysohn's Lemma.*) Let (E,d) be a metric space. For any subset $A \subset E$ and any point $x \in E$, the *distance* between x and A is defined by

$$d(x,A) = \inf_{a \in A} d(x,a).$$

(a) Verify that d is well-defined and calculate $d(x,A)$ when $x \in A$.

(b) Show that $d(x,A) = d(x,\bar{A})$, where $\bar{A}$ is the closure of A.

(c) Show that $d(\cdot, A)$ is 1-Lipschitz, that is,

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

for any $x, y \in E$.

(d) Let A and B be disjoint closed subsets of E. Prove the existence of a continuous function $f : E \longrightarrow \mathbf{R}$ such that:

(i) $0 \leq f(x) \leq 1$ for all $x \in E$;

(ii) $f(x) = 0$ for all $x \in A$;

(iii) $f(x) = 1$ for all $x \in B$.

Hint: consider an appropriate combination of $d(\cdot, A)$ *and* $d(\cdot, B)$.

Solution. **(a)** The infimum is taken over a non-empty set of non-negative numbers, all of which are 0 if $x \in A$.

(b) Observe that $A \subset \bar{A}$ so $d(x, A) \geq d(x, \bar{A})$. For the other inequality, consider α in $\bar{A}$. There exists a sequence $\{a_n\} \in A^{\mathbf{N}}$ that converges to α. Given x fixed, the function $d(x, \cdot)$ is continuous so $\lim_{n\to\infty} d(x, a_n) = d(x, \alpha)$. Since $d(x, a_n) \geq d(x, A)$ for every n, it follows that $d(x, \alpha) \geq d(x, A)$. This is true for every α in $\bar{A}$ so $d(x, \bar{A}) \geq d(x, A)$.

(c) For $x, y \in E$ and $a \in A$, the triangle inequality and the definition of $d(x, A)$ imply that $d(x, A) \leq d(x, y) + d(y, a)$. This is true for every $a \in A$ so $d(x, A) \leq d(x, y) + d(y, A)$ and we get $d(x, A) - d(y, A) \leq d(x, y)$. The same argument gives $d(y, A) - d(x, A) \leq d(x, y)$ hence the result.

(d) Consider $x \longmapsto \dfrac{d(x, A)}{d(x, A) + d(x, B)}$. This is (a special case of) Urysohn's Lemma.

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6. (*Separation of pseudo-metric spaces.*) A *pseudo-metric* on a set E is a map $d : E \times E \longrightarrow \mathbf{R}_+$ satisfying

(i) $d(x, y) = d(y, x)$

(ii) $x = y \quad \Rightarrow \quad d(x, y) = 0$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

for all x, y, z in E.

(a) Check that the relation $\sim$ defined on E by:

$$x \sim y \quad \Leftrightarrow \quad d(x, y) = 0$$

is an equivalence relation.

Denote by $\tilde{x}$ the class of $x \in E$ for this relation, and by $\tilde{E}$ the quotient $E/\sim$.

(b) Verify that the map $\tilde{d} : (\tilde{x}, \tilde{y}) \longmapsto d(x, y)$ is a well-defined metric on $\tilde{E}$.

Solution. **(a)** Symmetry follows from (i), reflexivity from (ii) and transitivity from (iii).

(b) To check that the definition of $\tilde{d}(\tilde{x}, \tilde{y})$ does not depend on the choice of representatives for $\tilde{x}$ and $\tilde{y}$, assume that $x \sim x'$ and $y \sim y'$. Then, the triangle inequality implies

$$d(x, y) \leq \underbrace{d(x, x')}_{=0} + d(x', y') + \underbrace{d(y, y')}_{=0},$$

hence $d(x, y) \leq d(x', y')$. The reverse inequality is proved similarly, showing that $\tilde{d}$ is well-defined.

To prove that it is a metric, observe that conditions (i), (ii) and (iii) hold automatically for $\tilde{d}$. Finally, $\tilde{d}(\tilde{x}, \tilde{y}) = 0$ implies $\tilde{x} = \tilde{y}$ by construction so $\tilde{d}$ is a metric on $\tilde{E}$.

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7. (*Ultrametric distances.*) Let E be a set equipped with a map $d : E \times E \longrightarrow \mathbf{R}_+$ satisfying

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$

(iii) $d(x, y) \leq \max\big(d(x, z)\,,\, d(z, y)\big)$

for all x, y, z in E.

(a) Verify that (E, d) is a metric space.

(b) Prove that if $d(x, z) \neq d(z, y)$, then (iii) is an equality. What can be said of triangles in E?

(c) Let $x \in E$ and $r > 0$. Prove that $B(x, r) = B(y, r)$ for any $y \in B(x, r)$.

Solution. **(a)** It suffices to prove the triangle inequality, which is a weaker condition than (iii), as a consequence of the identity $\max(a, b) = a + b - \min(a, b)$ with a and b non-negative.

(b) Assume $d(x, z) < d(z, y)$ so that (iii) becomes $d(x, y) \leq d(z, y)$ and apply (iii) again:

$$d(z, y) \leq \max\big(d(z, x)\,,\, d(x, y)\big)$$

and the assumption implies that the right-hand side is $d(x, y)$, showing that $d(z, y) = d(x, y)$.

All triangles in an ultrametric space are isosceles, with the unequal side shorter than the other two.

(c) Let z be an element of $B(x, r)$. Then

$$d(y, z) \leq \max\big(\underbrace{d(y, x)}_{<r}\,,\, \underbrace{d(x, z)}_{<r}\big) < r,$$

showing that $B(x,r) \subset B(y,r)$. Conversely, consider z in $B(y,r)$. Then,

$$d(x,z) \leq \max\big(\underbrace{d(x,y)}_{<r}, \underbrace{d(y,z)}_{<r}\big) < r,$$

establishing that $B(x,r) = B(y,r)$.

In an ultrametric space, every point of a ball is a center.

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8. (*p-adic distance.*) Let p be a prime number. For $n \in \mathbf{Z} \setminus \{0\}$, denote by $\nu_p(n)$ the exponent of p in the prime factorization of n.

(a) Prove that the map d_p defined on $\mathbf{Z} \times \mathbf{Z}$ by

$$d_p(x,y) = \begin{cases} p^{-\nu_p(x-y)} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

satisfies the conditions of the previous problem.

(b) Determine $B(x, p^{-n})$ and $B_c(x, p^{-n})$ for $x \in \mathbf{Z}$ and $n \in \mathbf{N}$.

(c) Study the convergence of the sequence $u_n = 6^n$ in $(\mathbf{Z}, d_2)$ and in $(\mathbf{Z}, d_5)$.

Solution. **(a)** For (i) note that the p-valuations of x and $-x$ are equal for any $x \in \mathbf{Z}$. (ii) is immediate since no power of p is equal to 0. To prove (iii) it suffices to verify that the p-adic valuation $|\cdot|_p$ defined by $|n|_p = p^{-\nu_p(n)}$ and $|0|_p = 0$ satisfies the identity

$$|m+n|_p \leq \max\big(|m|_p\,,\,|n|_p\big).$$

Notice that $\nu_p(m+n) \geq \min\big(\nu_p(m)\,,\,\nu_p(n)\big)$, with equality unless $m+n=0$. The function $t \mapsto p^{-t}$ being decreasing, it follows that

$$|m+n|_p \leq p^{-\min\big(\nu_p(m)\,,\,\nu_p(n)\big)} \leq \max\big(p^{-\nu_p(m)}\,,\,p^{-\nu_p(n)}\big) = \max\big(|m|_p\,,\,|n|_p\big).$$

(b) For $y \in \mathbf{Z}$, the condition $d_p(x,y) < p^{-n}$ is equivalent to $\nu_p(x-y) < n$, meaning that $y \equiv x\ [p^{n+1}]$ so that

$$B(x, p^{-n}) = \big\{x + kp^{n+1}\,,\ k \in \mathbf{Z}\big\}\,.$$

A similar argument shows that

$$B_c(x, p^{-n}) = \{x + kp^n\,,\ k \in \mathbf{Z}\}\,.$$

Note that every closed ball is also an open ball (with a different radius).

(c) Note that $\nu_2(6^n) = n$. Therefore, $d_2(u_n, 0) = 2^{-n} \underset{n\to\infty}{\longrightarrow} 0$ and u_n converges to 0 in the 2-adic distance.

We will prove that the sequence is not Cauchy in the 5-adic distance: for any integers $p > q$,

$$(\dagger) \qquad \nu_5(u_p - u_q) = \nu_5\big(6^q(6^{p-q} - 1)\big) = \nu_5(6^{p-q} - 1),$$

so that $d_5(u_n, u_{n+1}) = \dfrac{1}{5}$ for any n, showing that $d_5(u_p, u_q)$ cannot be made arbitrarily small for p and q large enough.

It follows from $(\dagger)$ that the only possible accumulation point of the sequence is 1. Noting that $\varphi(5^n) = 5^n - 5^{n-1}$, an application of Fermat's little theorem shows that the subsequence $6^{(5^n)}$ is convergent.