Metric Topology

Due Feb. 10

Problem 1. Let (E_1, d_1) and (E_2, d_2) be metric spaces. Prove that the map defined by

$$d\big((x_1, x_2)\,,\,(y_1, y_2)\big) = \max\big(d_1(x_1, y_1)\,,\,d_2(x_2, y_2)\big)$$

is a metric on $E_1 \times E_2$.

Problem 2. Recall that a subset X of a topological space E is said *compact* if any open cover of X admits a finite subcover. In this problem, we assume that E is a metric space.

(a) What is a *totally bounded* set?

(b) What is a *sequentially compact* set?

(c) State all the implications between total boundedness, sequential compactness and compactness for a general metric space.

(d) State the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem for $\mathbf{R}^n$ equipped with its ordinary (Euclidean) metric.

(e) What parts of these results hold in greater generality?

See Section 9.5 of [Royden-Fitzpatrick] for inspiration.

Problem 3. Recall that a subset C of a linear space is said *convex* if for any x, y in C the line segment

$$[x, y] = \{(1-t)x + ty\,,\,0 \le t \le 1\}$$

is included in C. Prove that balls in a normed linear space are always convex.

Problem 4. Consider the map δ defined on $\mathbf{R}^2 \times \mathbf{R}^2$ by

$$\delta(u,v) = \begin{cases} \|u-v\| & \text{if } u \text{ and } v \text{ are colinear} \\ \|u\|+\|v\| & \text{otherwise} \end{cases} .$$

(a) Prove that δ is a distance.

(b) Describe geometrically the ball $B(u,r)$ for $u \in \mathbf{R}^2$ and $r > 0$.

(c) Is there a norm N on $\mathbf{R}^2$ such that $\delta(u,v) = N(u-v)$ for all $u, v \in \mathbf{R}^2$?

Problem 5. Let (E,d) be a metric space. For any subset $A \subset E$ and any point $x \in E$, the *distance* between x and A is defined by

$$d(x,A) = \inf_{a \in A} d(x,a).$$

(a) Verify that d is well-defined and calculate $d(x,A)$ when $x \in A$.

(b) Show that $d(x,A) = d(x,\bar{A})$, where $\bar{A}$ is the closure of A.

(c) Show that $d(\cdot, A)$ is 1-Lipschitz, that is,

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

for any $x, y \in E$.

(d) Let A and B be disjoint closed subsets of E. Prove the existence of a continuous function $f : E \longrightarrow \mathbf{R}$ such that:

(i) $0 \leq f(x) \leq 1$ for all $x \in E$;

(ii) $f(x) = 0$ for all $x \in A$;

(iii) $f(x) = 1$ for all $x \in B$.

Hint: consider an appropriate combination of $d(\cdot, A)$ *and* $d(\cdot, B)$.

Problem 6. A *pseudo-metric* on a set E is a map $d : E \times E \longrightarrow \mathbf{R}_+$ satisfying

(i) $d(x,y) = d(y,x)$

(ii) $x = y \quad \Rightarrow \quad d(x,y) = 0$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$

for all x, y, z in E.

(a) Check that the relation $\sim$ defined on E by:

$$x \sim y \quad \Leftrightarrow \quad d(x,y) = 0$$

is an equivalence relation.

Denote by $\tilde{x}$ the class of $x \in E$ for this relation, and by $\tilde{E}$ the quotient $E/\sim$.

(b) Verify that the map $\tilde{d} : (\tilde{x}, \tilde{y}) \longmapsto d(x,y)$ is a well-defined metric on $\tilde{E}$.

Problem 7. Let E be a set equipped with a map $d : E \times E \longrightarrow \mathbf{R}_+$ satisfying

(i) $d(x,y) = d(y,x)$

(ii) $d(x,y) = 0 \quad \Leftrightarrow \quad x = y$

(iii) $d(x,y) \leq \max\big(d(x,z)\,,\, d(z,y)\big)$

for all x, y, z in E.

(a) Verify that (E,d) is a metric space.

(b) Prove that if $d(x,z) \neq d(z,y)$, then (iii) is an equality. What can be said of triangles in E?

(c) Let $x \in E$ and $r > 0$. Prove that $B(x,r) = B(y,r)$ for any $y \in B(x,r)$.

Problem 8. (*Optional.*) Let p be a prime number. For $n \in \mathbf{Z} \setminus \{0\}$, denote by $\nu_p(n)$ the exponent of p in the prime factorization of n.

(a) Prove that the map d_p defined on $\mathbf{Z} \times \mathbf{Z}$ by

$$d_p(x,y) = \begin{cases} p^{-\nu_p(x-y)} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

satisfies the conditions of the previous problem.

(b) Determine $B(x, p^{-n})$ and $B_c(x, p^{-n})$ for $x \in \mathbf{Z}$ and $n \in \mathbf{N}$.

(c) Study the convergence of the sequence $u_n = 6^n$ in $(\mathbf{Z}, d_2)$ and in $(\mathbf{Z}, d_5)$.