Warm-up: Topology of the Real Line

Elements of Solution

Problem 1. Which of the following subsets of $\mathbf{R}$ are open?

$$A = (\pi, 5) \quad , \quad B = (1, \infty) \quad , \quad C = \left[\sqrt{2}, \sqrt{3}\right) \quad , \quad \mathbf{Q}.$$

Solution. Any interval of the form (a, b) is open: for $a < x < b$, let $r = \min\{x-a, b-x\}$ and notice that $(x-r, x+r) \subset (a, b)$. The argument can be adapted to the case when a or b is infinite, so that A and B are open.

Every open interval centered at $\sqrt{2}$ contains numbers $< \sqrt{2}$, hence cannot be included in C, which is therefore not open.

Irrationals are dense in $\mathbf{R}$ so that no open interval centered at a rational can contain only rationals, so that $\mathbf{Q}$ is not open.

Problem 2. Prove that unions and finite intersections of open sets are open.

Solution. The case of unions is immediate. Assume that $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are open. If their intersection is empty, there is nothing to check. If it is not, for every x in $\mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$, there exist positive numbers $r_1, \ldots, r_n$ such that $(x - r_i, x + r_i)$ is included in $\mathcal{U}_i$ for each $i \in \{1, \ldots, n\}$. Then $r = \min\{r_1, \ldots, r_n\}$ is positive and $(x-r, x+r)$ is contained in $\mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$, which is therefore open.

Problem 3. Show that an arbitrary intersection of open sets is not necessarily open.

Solution. The intervals $\left(1 - \dfrac{1}{n}, 2 + \dfrac{1}{n}\right)$ for $n > 0$ are open but their intersection, $[1, 2]$, is not. The argument given in the previous question fails to extend because the infimum of an infinite family of positive numbers may be 0.

Problem 4. Prove that a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous if and only if $f^{-1}(\mathcal{U})$ is open for any $\mathcal{U}$ open in $\mathbf{R}$.

Solution. Assume f continuous. Let $\mathcal{U}$ be an open subset of $\mathbf{R}$ and $x \in f^{-1}(\mathcal{U})$. Then $f(x)$ belongs to $\mathcal{U}$ open, so there exists $\varepsilon > 0$ such that

$$\big(f(x) - \varepsilon, f(x) + \varepsilon\big) \subset \mathcal{U}.$$

By continuity of f at x, there exists $\delta > 0$ such that

$$f\big((x-\delta, x+\delta)\big) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \subset \mathcal{U},$$

showing that $(x - \delta, x + \delta)$ is included in $f^{-1}(\mathcal{U})$, which is therefore open.

Conversely, if the inverse image of the open set $\big(f(x) - \varepsilon, f(x) + \varepsilon\big)$ is open, since it contains x, there must exist $\delta > 0$ such that

$$(x - \delta, x + \delta) \subset f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big),$$

showing that f is continuous at x.

Problem 5. Prove that a subset X of $\mathbf{R}$ is closed if and only if for any sequence $\{x_n\}_{n\in\mathbf{N}}$ of points of X that converges to a limit in $\mathbf{R}$, the limit is also in X.

Solution. Assume that X is closed and that $\{x_n\}_{n\in\mathbf{N}}$ is a sequence of elements of X that converges to a limit x in $\mathbf{R}$. If x is not in X, it must belong to its complement, which is open. Therefore, there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap X = \varnothing.$$

In particular, no term of the sequence can be found within ε of x, which contradicts the assumption that $\lim x_n = x$. It follows that $x \in X$.

Conversely, assume that convergent sequences of points of X have their limits in X. To check that X is closed, we will prove that if $y \notin X$, there exists $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon)$ is included in the complement of X. Indeed, if it was not the case, the intersection

$$\left(y - \frac{1}{n}, y + \frac{1}{n}\right) \cap X$$

would be non-empty for every positive integer n. Picking an element x_n is each of these sets would provide a sequence in X converging to y, hence a contradiction.

Problem 6. Find a subset of $\mathbf{R}$ that is neither open nor closed.

Solution. Any interval of the form $[a, b)$ works. So does $\mathbf{Q}$.

Problem 7. Is the set $X = \left[\sqrt{2}, \sqrt{3}\right) \cup \left(\sqrt{5}, \sqrt{6}\right]$ compact?

Solution. No: consider the family of open sets

$$\mathcal{U}_n = \left(1, \sqrt{3} - \frac{1}{n}\right) \cup \left(\sqrt{5} + \frac{1}{n}, 3\right)$$

for $n \geq 1$. The fact that the sequences $\sqrt{3} - \dfrac{1}{n}$ and $\sqrt{5} + \dfrac{1}{n}$ respectively converge to $\sqrt{3}$ from below and $\sqrt{5}$ from above guarantee that $\{\mathcal{U}_n\}_{n \geq 1}$ covers X.

Since the family is increasing, the union of any finite subfamily is of the form $\mathcal{U}_p$ for some $p \geq 1$, hence cannot cover X, which is therefore not compact.

Problem 8. Let K be a compact subset of $\mathbf{R}$ and $f : \mathbf{R} \longrightarrow \mathbf{R}$ a continuous function. Prove that $f(K)$ is compact.

Solution. Let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Since f is continuous, the family $\{f^{-1}(\mathcal{U}_\alpha)\}_{\alpha \in I}$ is an open cover of K, compact, so one can extract a finite subcover

$$\left\{f^{-1}(\mathcal{U}_{\alpha_1}), \ldots, f^{-1}(\mathcal{U}_{\alpha_p})\right\}$$

and $\left\{\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_p}\right\}$ is a finite subcover of $f(K)$, which is therefore compact.

Problem 9. Assume that K is a compact subset of $\mathbf{R}$. Prove that K is bounded.

Solution. Let x be a fixed element in K. The family of open intervals $(x - n, x + n)$ with $n \in \mathbf{N}$ covers K. The union of any finite subfamily is of the form $(x - N, x + N)$ for some $N \in \mathbf{N}$. If K is contained in such an interval, then it is included in the interval $(-|x| - N, |x| + N)$, hence bounded.

Alternatively, the family of open intervals $\{I_x\}_{x \in K}$ where $I_x = (x - 1, x + 1)$ covers K. If it admits a finite subcover, say $\{I_{x_1}, \ldots, I_{x_p}\}$, then K is included in the interval

$$(\min\{x_1, \ldots, x_p\} - 1, \max\{x_1, \ldots, x_p\} + 1)$$

hence bounded.

Problem 10. Prove that compact subsets of $\mathbf{R}$ are closed.

Solution. Assume that $K \subset \mathbf{R}$ is compact. To prove that its complement is open, consider $x_0 \notin K$ and let $r_k = |k - x_0|/2$ for every $k \in K$. The open intervals

$$\mathcal{U}_k = (k - r_k, k + r_k) \quad \text{and} \quad \mathcal{V}_k = (x_0 - r_k, x_0 + r_k)$$

are disjoint for every $k \in K$ and the family $\{\mathcal{U}_k\}_{k \in K}$ covers K. Extract a finite subcover $\{\mathcal{U}_{k_1}, \ldots, \mathcal{U}_{k_p}\}$:

$$K \subset \mathcal{U}_{k_1} \cup \ldots \cup \mathcal{U}_{k_p}$$

and consider the intersection of the corresponding sets $\mathcal{V}_{k_1}, \ldots, \mathcal{V}_{k_p}$. By construction, $\mathcal{V}_{k_1} \cap \ldots \cap \mathcal{V}_{k_p}$ is disjoint from $\mathcal{U}_{k_1} \cup \ldots \cup \mathcal{U}_{k_p}$ hence is included in the complement of K, which therefore contains $(x_0 - r, x_0 + r)$ where $r = \min\{r_{k_1}, \ldots, r_{k_p}\} > 0$.