Warm-up: Topology of the Real Line

Due Feb. 3

Problem 1. Which of the following subsets of $\mathbf{R}$ are open?

$$A = \left(\pi, \sqrt{11}\right) \quad , \quad B = (1, \infty) \quad , \quad C = \left[\sqrt{2}, \sqrt{3}\right) \quad , \quad \mathbf{Q}.$$

Problem 2. Prove that unions and finite intersections of open sets are open.

Problem 3. Show that an arbitrary intersection of open sets is not necessarily open.

Problem 4. Prove that a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous if and only if $f^{-1}(\mathcal{U})$ is open for any $\mathcal{U}$ open in $\mathbf{R}$.

Problem 5. Prove that a subset X of $\mathbf{R}$ is closed if and only if for any sequence $\{x_n\}_{n\in\mathbf{N}}$ of points of X that converges to a limit in $\mathbf{R}$, the limit is also in X.

Problem 6. Find a subset of $\mathbf{R}$ that is neither open nor closed.

Problem 7. Is the set $X = \left[\sqrt{2}, \sqrt{3}\,\right) \cup \left(\sqrt{5}, \sqrt{6}\,\right]$ compact?

Problem 8. Let K be a compact subset of $\mathbf{R}$ and $f : \mathbf{R} \longrightarrow \mathbf{R}$ a continuous function. Prove that $f(K)$ is compact.

Problem 9. Assume that K is a compact subset of $\mathbf{R}$. Prove that K is bounded.

Problem 10. Prove that compact subsets of $\mathbf{R}$ are closed.