Chapter 8

Root Space Decompositions

Henceforth we will focus on complex simple and semisimple Lie algebras. These algebras have an incredibly rich structure, all brought to you by Theorem 7.3.12. As for *real* semisimple Lie algebras, it turns out that the theory of such algebras essentially amounts to finding the real forms of complex semisimple Lie algebras. Unfortunately (really!), the study of such algebras is beyond the scope of this course.

8.1 Cartan Subalgebras

We begin with a generalized binomial expansion for derivations.

Exercise 8.1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and let $D \in \operatorname{Der}\mathfrak{g}$. Prove that for any x, $y \in \mathfrak{g}$, $\lambda, \mu \in \mathbb{F}$, and $k \in \mathbb{N}$,

$$(D - (\lambda + \mu)\, I_{\mathfrak{g}})^k\, [x, y] = \sum_{r=0}^{k} \binom{k}{r} [(D - \lambda\, I_{\mathfrak{g}})^r\, x, (D - \mu\, I_{\mathfrak{g}})^{k-r}\, y]$$

Lemma 8.1.2. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Then* $\operatorname{Der}\mathfrak{g}$ *contains the semisimple and nilpotent parts of all its elements.*

Proof. Let $D \in \operatorname{Der}\mathfrak{g}$, and let $D = S+N$ be its Jordan-Chevalley decomposition. Here S and N are commuting semisimple and nilpotent linear operators on $\mathfrak{g}$, respectively.

Since our field is $\mathbb{C}$, we can, by Theorem 1.7.7, decompose $\mathfrak{g}$ into a direct sum of generalized eigenspaces of D:

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_{\lambda}, \tag{8.1}$$

where the sum is taken over all eigenvalues of D, and

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \,|\, (D - \lambda\, I_\mathfrak{g})^k\, x = 0 \text{ for some } k \in \mathbb{Z}^+\}$$

We now claim that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Here $\mathfrak{g}_{\lambda+\mu}$ is understood to be $\{0\}$ if $\lambda + \mu$ is not an eigenvalue of D.

To prove the claim, we first note that by the remark after Theorem 1.7.1, $\mathfrak{g}_\lambda = \ker(D - \lambda\, I_\mathfrak{g})^n$, where $n = \dim \mathfrak{g}$. Suppose that $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$. Then by Exercise 8.1.1,

$$(D - (\lambda + \mu)\, I_\mathfrak{g})^{2n}\, [x, y] = \sum_{r=0}^{2n} \binom{2n}{r} [(D - \lambda\, I_\mathfrak{g})^r\, x, (D - \mu\, I_\mathfrak{g})^{2n-r}\, y].$$

At least one of the factors in each Lie bracket on the right hand side above vanishes. Hence the right hand side vanishes, and $[x, y]$ is annihilated by a power of $D - (\lambda + \mu)\, I_\mathfrak{g}$. Thus, $[x, y]$ belongs to the generalized eigenspace $\mathfrak{g}_{\lambda+\mu}$.

Assuming still that $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$, it is now easy to show that the semisimple operator S satisfies Leibniz' rule on $[x, y]$:

$$\begin{aligned} S[x, y] &= (\lambda + \mu)\, [x, y] \qquad (\text{since } [x, y] \in \mathfrak{g}_{\lambda+\mu}) \\ &= [\lambda x, y] + [x, \mu y] \\ &= [Sx, y] + [x, Sy]. \end{aligned}$$

If x and y are arbitrary elements of $\mathfrak{g}$, then we have $x = \sum_\lambda x_\lambda$ and $y = \sum_\lambda y_\lambda$, where $x_\lambda,\, y_\lambda \in \mathfrak{g}_\lambda$. Hence

$$\begin{aligned} S[x, y] &= S\left[\sum_\lambda x_\lambda\,,\, \sum_\mu y_\mu\right] \\ &= S\left(\sum_{\lambda,\mu} [x_\lambda, y_\mu]\right) \\ &= \sum_{\lambda,\mu} ([Sx_\lambda, y_\mu] + [x_\lambda, Sy_\mu]) \\ &= \left[S\left(\sum_\lambda x_\lambda\right), \sum_\mu y_\mu\right] + \left[\sum_\lambda x_\lambda, S\left(\sum_\mu y_\mu\right)\right] \\ &= [Sx, y] + [x, Sy] \end{aligned}$$

Hence Leibniz' rule holds for S on $\mathfrak{g}$, and so $S \in \operatorname{Der} \mathfrak{g}$. It follows that $N = D - S \in \operatorname{Der} \mathfrak{g}$. □

Now if $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$, then we know by Theorem 7.1.5 that $\mathfrak{g}$ is complete; that is, $\operatorname{ad} \mathfrak{g} = \operatorname{Der} \mathfrak{g}$. Lemma 8.1.2 then implies that if

$x \in \mathfrak{g}$, the semisimple and nilpotent parts of $\operatorname{ad} x$ belong to $\operatorname{ad} \mathfrak{g}$. Since the map $\operatorname{ad} : \mathfrak{g} \to \operatorname{ad} \mathfrak{g}$ is injective, there must therefore exist *unique* elements x_s and x_n in $\mathfrak{g}$ such that

$$\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$$

is the Jordan-Chevalley decomposition of $\operatorname{ad} x$. Here $\operatorname{ad} x_s$ is semisimple and $\operatorname{ad} x_n$ is nilpotent. From the above, we obtain $\operatorname{ad} x = \operatorname{ad}(x_s + x_n)$, and so

$$x = x_s + x_n. \tag{8.2}$$

Definition 8.1.3. Equation 8.2 is called the *abstract Jordan-Chevalley decomposition* of $x \in \mathfrak{g}$. x_s is called the *semisimple part* of x, and x_n is called the *nilpotent part* of x.

Definition 8.1.3 begs the question: what if $\mathfrak{g} \subset \mathfrak{gl}(V)$ for some complex vector space V? Does the abstract Jordan decomposition of any $X \in \mathfrak{g}$ coincide with its usual one? The answer is yes, as the following exercise shows.

First note that it is an immediate consequence of Lemma 6.3.1 and Lemma 1.9.9 (and its nilpotent counterpart) that if N is a nilpotent element of $\mathfrak{gl}(V)$, then so is $\operatorname{ad} N$, and likewise, if S is a semisimple element of $\mathfrak{gl}(V)$, then so is $\operatorname{ad} S$.

Exercise 8.1.4. Let V be a complex vector space, and let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Let $X \in \mathfrak{g}$, and let $X = S + N$ be its Jordan-Chevalley decomposition, and $X = X_s + X_n$ its abstract Jordan-Chevalley decomposition. Using the following steps, prove that $S = X_s$ and $N = X_n$.

(a). We know that X_s and X_n belong to $\mathfrak{g}$. The problem is that S or N may not belong to $\mathfrak{g}$. Prove that S and N normalize $\mathfrak{g}$; that is, $[S, \mathfrak{g}] \subset \mathfrak{g}$ and $[N, \mathfrak{g}] \subset \mathfrak{g}$.

(b). Prove that $S - X_s$ and $N - X_n$ both belong to the centralizer of $\mathfrak{g}$ in $\mathfrak{gl}(V)$.

(c). Prove that S commutes with X_s and N commutes with X_n.

(d). Since $\mathfrak{g}$ is semisimple, Weyl's theorem and Theorem 7.3.8 imply that V decomposes into a direct sum of $\mathfrak{g}$-invariant irreducible subspaces $V = V_1 \oplus \cdots \oplus V_m$. Prove that each subspace V_i is invariant under S and N.

(e). Since X_s, $X_n \in \mathfrak{g}$, each V_i is also X_s- and X_n-invariant. Show that $S - X_s$ and $N - X_n$ are scalar operators on each V_i. That is, prove that for each i, there is a $\lambda_i \in \mathbb{C}$ such that $(S - X_s)|_{V_i} = \lambda_i \, I_{V_i}$, and similarly for $N - X_n$.

(f). Prove that X_s is a semisimple linear operator on each V_i, and hence on V.

(g). Prove that for each $Y \in \mathfrak{g}$, $\operatorname{tr} Y|_{V_i} = 0$. Hence $\operatorname{tr} X|_{V_i} = \operatorname{tr} X_s|_{V_i} = \operatorname{tr} X_n|_{V_i} = 0$.

(h). Prove that $N = X_n$ and that $S = X_s$.

Example 8.1.5. In the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, it is a lot easier to prove that the abstract and the regular Jordan-Chevalley decompositions coincide. Let $X \in \mathfrak{sl}(n, \mathbb{C})$, and let $X = S + N$ be its regular Jordan-Chevalley decomposition. Since N is nilpotent, we have $\operatorname{tr} N = 0$. Hence $N \in \mathfrak{sl}(n, \mathbb{C})$, and so $S = X - N \in \mathfrak{sl}(n, \mathbb{C})$. Since $\operatorname{ad} S$ and $\operatorname{ad} N$ are semisimple and nilpotent, respectively, on $\mathfrak{gl}(n, \mathbb{C})$, they are likewise semisimple and nilpotent, respectively, on $\mathfrak{sl}(n, \mathbb{C})$. Hence $S = X_s$ and $N = X_n$.

Definition 8.1.6. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. A *toral subalgebra* of $\mathfrak{g}$ is a subalgebra consisting entirely of semisimple elements x. (This means $x = x_s$ in Definition 8.1.3.)

If $\mathfrak{g} \neq \{0\}$, does $\mathfrak{g}$ have any nonzero toral subalgebras? Yes! The reason is that there is at least one element x of $\mathfrak{g}$ such that $x_s \neq 0$. Otherwise, $x = x_n$ for all $x \in \mathfrak{g}$, so $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$, and hence by Engel's theorem (Theorem 5.2.1), $\mathfrak{g}$ is nilpotent. But no nilpotent Lie algebra is semisimple. (Why?) So choose $x \in \mathfrak{g}$ such that $x_s \neq 0$. Then $\mathfrak{a} = \mathbb{C}x_s$ is a one-dimensional toral subalgebra of $\mathfrak{g}$. Note that because $\mathfrak{a}$ is one-dimensional, it is obviously abelian.

Surprisingly, it turns out that all toral subalgebras are abelian.

Proposition 8.1.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Then any toral subalgebra $\mathcal{T}$ of $\mathfrak{g}$ is abelian.*

Proof. Let $x \in \mathcal{T}$. Since $\mathcal{T}$ is a subalgebra, $\mathcal{T}$ is $\operatorname{ad} x$-invariant. We want to show that $\operatorname{ad} x|_{\mathcal{T}} = 0$, since this will obviously imply that $\mathcal{T}$ is abelian. Now clearly, $\operatorname{ad} x = 0$ for $x = 0$, so let us assume that $x \neq 0$.

Now since $\operatorname{ad} x$ is semisimple, its restriction $\operatorname{ad} x|_{\mathcal{T}}$ is also semisimple, by Lemma 1.9.9. Thus it suffices to prove that any eigenvalue α of $\operatorname{ad} x|_{\mathcal{T}}$ is 0. So suppose that α is an eigenvalue of $\operatorname{ad} x|_{\mathcal{T}}$ and that $y \in \mathcal{T}$ is an eigenvector corresponding to α. Then $[x, y] = \operatorname{ad} x\,(y) = \alpha\, y$. Hence

$$\begin{aligned} \operatorname{ad} y\,[y, x] &= -\operatorname{ad} y\,[x, y] \\ &= -\operatorname{ad} y\,(\alpha\, y) \\ &= -\alpha\,[y, y] \\ &= 0. \end{aligned} \tag{8.3}$$

Now $y \in \mathcal{T}$, so $\operatorname{ad} y|_{\mathcal{T}}$ is semisimple. Thus, there is a basis $(e_1, \ldots, e_k)$ of $\mathcal{T}$ consisting of eigenvectors of $\operatorname{ad} y|_{\mathcal{T}}$, with respective eigenvalues $\alpha_1, \ldots, \alpha_k$.

We have, of course, $x = \sum_{i=1}^{k} \lambda_i\, e_i$, for some scalars λ_i, not all 0. Then

$\operatorname{ad} y\,(x) = [y,x] = \sum_{i=1}^k \lambda_i\,[y,e_i] = \sum_{i=1}^k \lambda_i\alpha_i\,e_i$, and so by equation (8.3),

$$\begin{aligned} 0 &= \operatorname{ad} y\,[y,x] \\ &= \operatorname{ad} y\,\left(\sum_{i=1}^k \lambda_i\alpha_i\,e_i\right) \\ &= \sum_{i=1}^k \lambda_i\alpha_i\,[y,e_i] \\ &= \sum_{i=1}^k \lambda_i\alpha_i^2\,e_i. \end{aligned}$$

It follows that $\lambda_i\alpha_i^2 = 0$ for all i, from which it follows that $\lambda_i\alpha_i = 0$ for all i. Hence $\alpha\,y = \operatorname{ad} x\,(y) = [x,y] = -[y,x] = -\sum_{i=1}^k \lambda_i\alpha_i\,e_i = 0$. Since $y \neq 0$, we conclude that $\alpha = 0$, proving the proposition. □

Definition 8.1.8. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* of $\mathfrak{g}$ is a *maximal* toral subalgebra of $\mathfrak{g}$; i,e., a toral subalgebra which is not properly contained in any other toral subalgebra of $\mathfrak{g}$.

Any toral subalgebra of maximal dimension is obviously a Cartan subalgebra of $\mathfrak{g}$. It is an important result in Lie theory that any two Cartan subalgebras are conjugate under an automorphism of $\mathfrak{g}$.

Theorem 8.1.9. *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$. Then there exists an automorphism φ of $\mathfrak{g}$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$.*

In particular, any two Cartan subalgebras of $\mathfrak{g}$ have the same dimension, which is called the *rank* of $\mathfrak{g}$.

Theorem 8.1.9 says that the behavior of any two Cartan subalgebras of $\mathfrak{g}$ with respect to the adjoint representation is exactly the same. Since the proof of Theorem 8.1.9 requires some Lie group theory, we will omit it.

8.2 Root Space Decomposition

In this section, we examine the root space structure of a complex semisimple Lie algebra $\mathfrak{g}$. Knowledge of this structure is absolutely vital for any further study of semisimple Lie theory.

Lemma 8.2.1. *Let $\mathfrak{h}$ be a toral subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Then there is a basis of $\mathfrak{g}$ relative to which the linear operators $\operatorname{ad} h$ all have diagonal matrices, for every $h \in \mathfrak{h}$.*

Proof. Let $h_1, \dots, h_l$ be a basis of $\mathfrak{h}$. then $\operatorname{ad} h_1, \dots, \operatorname{ad} h_k$ are semisimple linear operators on $\mathfrak{g}$. Moreover, these operators commute, since $[\operatorname{ad} h_i, \operatorname{ad} h_j] = \operatorname{ad}[h_i, h_j] = 0$. Hence by Exercise 1.9.12, there is a basis of $\mathfrak{g}$ relative to which each $\operatorname{ad} h_i$ has a diagonal matrix. If $h \in \mathfrak{h}$, then $\operatorname{ad} h$ is a linear combination of the $\operatorname{ad} h_i$, so the matrix of $\operatorname{ad} h$ relative to this basis is also diagonal. □

The basis obtained in Lemma 8.2.1 thus consists of joint eigenvectors of all the elements of $\operatorname{ad} \mathfrak{h}$.

Let us now assume that $\mathfrak{h}$ is a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Choose a basis of $\mathfrak{g}$ consisting of joint eigenvectors of $\operatorname{ad} h$. If v is an element of this basis, then for every $h \in \mathfrak{h}$, we have

$$\operatorname{ad} h\,(v) = \alpha(h)\, v, \tag{8.4}$$

where the complex coefficient $\alpha(h)$ clearly depends on $\mathfrak{h}$. The mapping $h \mapsto \alpha(h)$ is easily seen to be a linear functional on $\mathfrak{h}$, and it is also clear that v belongs to the *joint eigenspace* corresponding to $\alpha \in \mathfrak{h}^*$.

For each $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha$ denote the joint eigenspace corresponding to α:

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [h, v] = \alpha(h)\, v \text{ for all } h \in \mathfrak{h}\} \tag{8.5}$$

Of course, for a given α, $\mathfrak{g}_\alpha$ could very well be $\{0\}$. Nonetheless, each vector in our basis belongs to a unique joint eigenspace, and we obtain the direct sum decomposition

$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha \tag{8.6}$$

where the sum ranges over all $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$.

The decomposition (8.6) is very important, and we will study its structure in some detail.

Observe that if 0 denotes the zero linear functional on $\mathfrak{h}$, then the joint eigenspace $\mathfrak{g}_0$ is the centralizer $\mathfrak{c}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is abelian, we have $\mathfrak{h} \subset \mathfrak{g}_0$, so in particular $\mathfrak{g}_0 \neq \{0\}$.

Definition 8.2.2. Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Any *nonzero* linear functional $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$ is called a *root of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$. If α is a root, the joint eigenspace $\mathfrak{g}_\alpha$ is called the root space corresponding to α. We denote the set of all roots by Δ.

We note that Δ is a *nonempty* finite set. If it was empty, then $\mathfrak{g}_0 = \mathfrak{c} =$ would coincide with $\mathfrak{g}$, making $\mathfrak{h}$ a non-zero subspace of the center of $\mathfrak{g}$, which we know is trivial.

Since $\mathfrak{g}$ is finite-dimensional and since each root space $\mathfrak{g}_\alpha$ has dimension ≥ 1, we see that the set Δ of roots is finite.

Definition 8.2.3. If $\mathfrak{h}$ is a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, then we can rewrite the decomposition (8.6) as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha. \tag{8.7}$$

We call (8.7) the *root space decomposition of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$.

For the rest of this section, we assume that $\mathfrak{h}$ is a fixed Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$.

Theorem 8.2.4. *The joint eigenspaces* $\mathfrak{g}_\alpha$ *satisfy the following properties:*

(a) For $\alpha,\, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(b) If $\alpha \neq 0$ *and* $x \in \mathfrak{g}_\alpha$, *then* $\operatorname{ad} x$ *is nilpotent.*

(c) If $\alpha,\, \beta \in \mathfrak{h}^*$ *such that* $\alpha + \beta \neq 0$, *then* $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$.

Proof. (a) Let $h \in \mathfrak{h}$, and let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$. Then, since $\operatorname{ad} h$ is a derivation, we have

$$\begin{aligned}\operatorname{ad} h\,[x,y] &= [\operatorname{ad} h\,(x), y] + [x, \operatorname{ad} h\,(y)]\\ &= [\alpha(h)\,x, y] + [x, \beta(h)\,y]\\ &= (\alpha(h)+\beta(h))\,[x,y]\\ &= (\alpha+\beta)(h)\,[x,y].\end{aligned}$$

This shows that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(b) We can assume that x is a nonzero element of $\mathfrak{g}_\alpha$. Let $\beta \in \mathfrak{h}^*$ such that $\mathfrak{g}_\beta \neq \{0\}$. Then by Part (a), $\operatorname{ad} x\,(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\beta+\alpha}$. Applying Part (a) again, we have $(\operatorname{ad} x)^2\,(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\beta+2\alpha}$. In general, for all $k \in \mathbb{Z}^+$,

$$(\operatorname{ad} x)^k\,(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\beta+k\alpha}$$

Now the set of $\beta \in \mathfrak{h}^*$ such that $\mathfrak{g}_\beta \neq \{0\}$ is finite, so, since $\alpha \neq 0$, there is a nonnegative integer $k = k_\beta$ such that $\mathfrak{g}_{\beta+k_\beta\alpha} = \{0\}$. Then of course $\mathfrak{g}_{\beta+k_\beta\,\alpha} = \{0\}$, and so

$$(\operatorname{ad} x)^{k_\beta}\;(\mathfrak{g}_\beta) = \{0\}.$$

Let $N = \max\{k_\beta\}$, where the maximum is taken over all β such that $\mathfrak{g}_\beta \neq \{0\}$. Then we have $(\operatorname{ad} x)^N = 0$, and hence $\operatorname{ad} x$ is nilpotent.

(c) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. For any $h \in \mathfrak{h}$, Lemma 6.4.1 implies that $B([x,h],y) = B(x,[h,y])$, hence:

$$\begin{aligned}0 &= B([h,x],y) + B(x,[h,y])\\ &= B(\alpha(h)\,x, y) + B(x, \beta(h)\,y)\\ &= (\alpha(h)+\beta(h))\,B(x,y).\end{aligned} \tag{8.8}$$

Since $\alpha+\beta \neq 0$, we may choose an element $h_0 \in \mathfrak{h}$ such that $\alpha(h_0)+\beta(h_0) \neq 0$. If we plug in $h = h_0$ in (8.8), we obtain $B(x, y) = 0$. Since x and y are arbitrary in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$, it follows that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$.

□

Theorem 8.2.4 implies, in particular, that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$, so that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$.

Corollary 8.2.5. *The Killing form B is nondegenerate on $\mathfrak{g}_0$.*

Proof. The assertion is that $B|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is nondegenerate. If $x \in \mathfrak{g}_0$, then by Theorem 8.2.4, we have $B(x, \mathfrak{g}_\alpha) = \{0\}$ for all $\alpha \neq 0$ in $\mathfrak{h}^*$. Suppose that $x \neq 0$. Then since B is nondegenerate, $B(x, \mathfrak{g}) \neq \{0\}$. Hence

$$\begin{aligned}\{0\} &\neq B(x, \mathfrak{g}) \\ &= B\big(x, \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\big) \\ &= B(x, \mathfrak{g}_0) + \sum_{\alpha\in\Delta} B(x, \mathfrak{g}_\alpha) \\ &= B(x, \mathfrak{g}_0).\end{aligned}$$

This shows that B is nondegenerate on $\mathfrak{g}_0$. □

Lemma 8.2.6. *The Killing form B is nondegenerate on $\mathfrak{h}$.*

Proof. Suppose that $h \in \mathfrak{h}$ satisfies $B(h, \mathfrak{h}) = \{0\}$. We will prove that $B(h, \mathfrak{g}_0) = \{0\}$, so that by Lemma 8.2.5, it will follow that $h = 0$.

Let x be any element of $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{h})$, and let $x = x_s + x_n$ be its abstract Jordan-Chevalley decomposition. Since $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ are the semisimple and nilpotent parts of $\operatorname{ad} x$, they are polynomials in $\operatorname{ad} x$ with zero constant term. (See Theorem 1.9.14.) Since $\operatorname{ad} x(\mathfrak{h}) = \{0\}$, we conclude that $\operatorname{ad} x_s(\mathfrak{h}) = \{0\}$ and $\operatorname{ad} x_n(\mathfrak{h}) = \{0\}$ as well. Thus $x_s, x_n \in \mathfrak{c}(\mathfrak{h}) = \mathfrak{g}_0$. This means that $[x_s, \mathfrak{h}] = [x_n, \mathfrak{h}] = \{0\}$.

Next we claim that $x_s \in \mathfrak{h}$. Since x_s is a semisimple element of $\mathfrak{g}$ commuting with $\mathfrak{h}$, the subspace $\mathbb{C}x_s + \mathfrak{h}$ is a toral subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Since $\mathfrak{h}$ is *maximal* toral, this implies that $\mathbb{C}x_s + \mathfrak{h} = \mathfrak{h}$, and therefore $x_s \in \mathfrak{h}$. In particular, by the hypothesis on h, we have $B(x_s, h) = 0$.

Now x_n commutes with h, so $\operatorname{ad} x_n$ commutes with $\operatorname{ad} h$. Since $\operatorname{ad} x_n$ is nilpotent, this must also be true of $\operatorname{ad} x_n \circ \operatorname{ad} h$: $(\operatorname{ad} x_n)^N = 0 \implies (\operatorname{ad} x_n \circ \operatorname{ad} h)^N = (\operatorname{ad} x_n)^N \circ (\operatorname{ad} h)^N = 0$. Hence $B(x_n, h) = \operatorname{tr}\,(\operatorname{ad} x_n \circ \operatorname{ad} h) = 0$.

Together with $B(x_s, h) = 0$, we conclude that $B(x, h) = B(x_n, h) + B(x_s, h) = 0$. Since x is arbitrary in $\mathfrak{g}_0$, we get $B(\mathfrak{g}_0, h) = \{0\}$, from which we obtain $h = 0$. □

Lemma 8.2.7. *Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{h})$. Then $\mathfrak{g}_0 = \mathfrak{h}$.*

Proof. We first repeat a key observation made during the proof of Lemma 8.2.6, namely, that if $x = x_s + x_n$ is the abstract Jordan decomposition of $x \in \mathfrak{g}_0$, then $x_s \in \mathfrak{h}$ and $x_n \in \mathfrak{g}_0$. The proof is then carried out by proving the following successive assertions:

Step 1: $\mathfrak{g}_0$ is nilpotent. By Engel's theorem, it suffices to prove that for each $x \in \mathfrak{g}_0$, $(\operatorname{ad} x)|_{\mathfrak{g}_0}$ is nilpotent. Let $x = x_s + x_n$ be the abstract Jordan decomposition of x. Since $x_s \in \mathfrak{h}$, we have $[x_s, \mathfrak{g}_0] = \{0\}$, and thus $(\operatorname{ad} x_s)|_{\mathfrak{g}_0} = 0$. On the other hand $\operatorname{ad} x_n$ is nilpotent on $\mathfrak{g}$, and so its restriction to $\mathfrak{g}_0$ is nilpotent. Therefore $(\operatorname{ad} x)|_{\mathfrak{g}_0} = (\operatorname{ad} x_n)|_{\mathfrak{g}_0}$ is nilpotent.

Step 2: $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = \{0\}$. Note that $B(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0]) = B([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = B(\{0\}, \mathfrak{g}_0) = \{0\}$. Thus if $h \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$, it follows that $B(\mathfrak{h}, h) \subset B(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0]) = \{0\}$. Since B is nondegenerate on $\mathfrak{h}$, this forces $h = 0$.

Step 3: $\mathfrak{g}_0$ is abelian. Assume that $[\mathfrak{g}_0, \mathfrak{g}_0] \neq \{0\}$. From the descending central series for the nilpotent Lie algebra $\mathfrak{g}_0$, it is clear that if $\mathfrak{c}_0$ denotes the center of $\mathfrak{g}_0$, then $\mathfrak{c}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0] \neq \{0\}$. Let c be a nonzero element of $\mathfrak{c}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$, and let $c = c_s + c_n$ be its abstract Jordan-Chevalley decomposition. If $c_n = 0$, then from the proof of Lemma 8.2.6, we get $0 \neq c = c_s \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = \{0\}$, a contradiction. Hence $c_n \neq 0$.

Again, from our key observation, we know that $c_n \in \mathfrak{g}_0$. Since $c \in \mathfrak{c}_0$ and $c_s \in \mathfrak{h} \subset \mathfrak{c}_0$, we see that $c_n \in \mathfrak{c}_0$. Thus, for all $x \in \mathfrak{g}_0$, $[c_n, x] = 0$ and therefore $[\operatorname{ad} c_n, \operatorname{ad} x] = 0$. Now $\operatorname{ad} c_n$ is a nilpotent linear operator, and we conclude that $\operatorname{ad} c_n \circ \operatorname{ad} x$ is also nilpotent for all $x \in \mathfrak{g}_0$. Hence, for all such x, $B(c_n, x) = \operatorname{tr}(\operatorname{ad} c_n \circ \operatorname{ad} x) = 0$. It follows that $B(c_n, \mathfrak{g}_0) = \{0\}$, contradicting the nondegeneracy of B on $\mathfrak{g}_0$.

This contradiction leads us to conclude that $[\mathfrak{g}_0, \mathfrak{g}_0] = \{0\}$.

Step 4: $\mathfrak{g}_0 = \mathfrak{h}$. Suppose that $x \in \mathfrak{g}_0 \setminus \mathfrak{h}$. If $x = x_s + x_n$ is its abstract Jordan-Chevalley decomposition, then we must have $x_n \neq 0$. Now $x_n \in \mathfrak{g}_0$, so, since $\mathfrak{g}_0$ is abelian, we conclude that $\operatorname{ad} x_n \circ \operatorname{ad} y$ is a nilpotent linear operator for all $y \in \mathfrak{g}_0$. Hence $B(x_n, y) = 0$ for all $y \in \mathfrak{g}_0$. Since B is nondegenerate on $\mathfrak{g}_0$, this forces $x_n = 0$, a contradiction. Thus $\mathfrak{g}_0 = \mathfrak{h}$, completing the proof of Lemma 8.2.7. □

Theorem 8.2.8. *(Root Space Decomposition) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a Cartan subalgebra. Then $\mathfrak{g}$ is a direct sum of $\mathfrak{h}$ and the root spaces of $\mathfrak{h}$:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{8.9}$$

Theorem 8.2.8 is an immediate consequence of equation (8.7) and Lemma 8.2.7.

Theorem 8.2.9. *Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Then for each $\varphi \in \mathfrak{h}^*$, there is a unique element h_φ such that $\varphi(h) = B(h_\varphi, h)$ for all $h \in \mathfrak{h}$.*

Proof. This follows immediately from Proposition 1.10.8. □

Note that according to Proposition 1.10.8, we also have $h_{\alpha\varphi+\beta\psi} = \alpha h_\varphi + \beta h_\psi$, for all φ, $\psi \in \mathfrak{h}^*$ and all $\alpha, \beta \in \mathbb{C}$.

Definition 8.2.10. We transfer the Killing form B to the dual space $\mathfrak{h}^*$ by $B(\phi, \psi) = B(h_\phi, h_\psi)$ for all ϕ, $\psi \in \mathfrak{h}^*$.

Using the convention in Definition 8.2.10, we have $\varphi(h_\psi) = B(h_\varphi, h_\psi) = \psi(h_\varphi)$. By Lemma 8.2.6, we also see that B is nondegenerate on $\mathfrak{h}^*$.

Let us now investigate the root spaces $\mathfrak{g}_\alpha$.

Theorem 8.2.11. *(Theorem on Roots) Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, and let Δ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then:*

(a) There are $\dim \mathfrak{h}$ *linearly independent roots which thus form a basis of $\mathfrak{h}^*$.*

(b) If α is a root, then so is $-\alpha$.

(c) If α is a root and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = B(x, y)\, h_\alpha$.

(d) If α is a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\, h_\alpha$.

(e) For each $\alpha \in \Delta$, $\alpha(h_\alpha) = B(\alpha, \alpha) \neq 0$.

(f) Let α be a root, and let $h^*_\alpha = 2h_\alpha/\alpha(h_\alpha)$. If e_α is a nonzero element of $\mathfrak{g}_\alpha$, then there is an $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha,\, f_\alpha,\, h^*_\alpha)$ is the basis of a three-dimensional simple Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Proof. (a) The claim here is that Δ spans the dual space $\mathfrak{h}^*$. Suppose, to the contrary, that Δ does *not* span $\mathfrak{h}^*$. Then there exists a nonzero vector $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. For each $x \in \mathfrak{g}_\alpha$, we therefore obtain $[h, x] = \alpha(h)\, x = 0$. Hence $[h, \mathfrak{g}_\alpha] = \{0\}$ for all $\alpha \in \Delta$. Since $\mathfrak{h}$ is abelian, we also have $[h, \mathfrak{h}] = \{0\}$. Hence by Theorem 8.2.8, $[h, \mathfrak{g}] = \{0\}$, and therefore h lies in the center $\mathfrak{c}$ of $\mathfrak{g}$. But since $\mathfrak{g}$ is semisimple, $\mathfrak{c} = \{0\}$, so $h = 0$, a contradiction.

(b) Suppose that α is a root, but that $-\alpha$ isn't. Let x be a nonzero element of $\mathfrak{g}_\alpha$. Then by Theorem 8.2.4 (c), $B(x, \mathfrak{g}_\beta) = \{0\}$ for all $\beta \in \Delta$ (including $\beta = \alpha$). For the same reason, we have $B(x, \mathfrak{g}_0) = B(x, \mathfrak{h}) = \{0\}$. Thus, by Theorem 8.2.8, we obtain $B(x, \mathfrak{g}) = \{0\}$. This contradicts the fact that B is nondegenerate on $\mathfrak{g}$.

(c) Suppose that α is a root and that $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Then according to Theorem 8.2.4 (a), $[x, y] \in \mathfrak{g}_0 = \mathfrak{h}$. We can determine the vector $[x, y]$ in $\mathfrak{h}$ as follows: For any $h \in \mathfrak{h}$, we have

$$\begin{aligned} B(h, [x, y]) &= B([h, x], y) \\ &= B(\alpha(h)\, x, y) \\ &= \alpha(h)\, B(x, y) \\ &= B(x, y)\, B(h, h_\alpha) \\ &= B(h, B(x, y)\, h_\alpha). \end{aligned}$$

Since B is nondegenerate on $\mathfrak{h}$ (Theorem 8.2.6), we see that $[x, y] = B(x, y)\, h_\alpha$.

(d) By part (c), it suffices to show that if $0 \neq x \in \mathfrak{g}_\alpha$, then there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $B(x, y) \neq 0$. Suppose, to the contrary, that $B(x, \mathfrak{g}_{-\alpha}) = \{0\}$. Now by Theorem 8.2.4 (c), $B(x, \mathfrak{g}_\beta) = \{0\}$ for all $\beta \in \Delta$ such that $\beta \neq -\alpha$, and likewise $B(x, \mathfrak{h}) = \{0\}$. Thus $B(x, \mathfrak{g}) = \{0\}$, contradicting the nondegeneracy of B on $\mathfrak{g}$.

(e) Let $\alpha \in \Delta$, and suppose that $\alpha(h_\alpha) = 0$. By part (d), it is possible to choose vectors $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $B(x, y) \neq 0$. Let $\mathfrak{s} = \mathbb{C}\, x + \mathbb{C}\, y + \mathbb{C}\, h_\alpha$. Then, since $[x, y] = c\, h_\alpha$ (with $c = B(x, y) \neq 0$) and $[h_\alpha, x] = \alpha(h_\alpha)\, x = 0$, $[h_\alpha, y] = \alpha(h_\alpha)\, y = 0$, we see that $\mathfrak{s}$ is a solvable Lie subalgebra of $\mathfrak{g}$ with derived algebra $\mathfrak{s}' = \mathbb{C}h_\alpha$. The algebra $\operatorname{ad}\mathfrak{s}$ is a solvable Lie algebra of linear operators on $\mathfrak{g}$, so by Lie's Theorem (Theorem 4.2.3), $\mathfrak{g}$ has a basis relative to which every element of $\operatorname{ad}\mathfrak{s}$ has an upper triangular matrix. With respect to this basis, the elements of $\operatorname{ad}\mathfrak{s}'$ have strictly upper triangular matrices. In particular, this means that $\operatorname{ad} h_\alpha$ has a strictly upper triangular matrix, and so $\operatorname{ad} h_\alpha$ is nilpotent. But $\operatorname{ad} h_\alpha$ is semisimple, so $\operatorname{ad} h_\alpha = 0$, whence $h_\alpha = 0$, and therefore $\alpha = 0$, contradicting the fact that the elements of Δ are nonzero.

(f) Let $0 \neq e_\alpha \in \mathfrak{g}_\alpha$. Then by part (d) above, there exists an element $f'_\alpha \in \mathfrak{g}_{-\alpha}$ such that $B(e_\alpha, f'_\alpha) = c \neq 0$. Let $f_\alpha = 2f'_\alpha/(c\,\alpha(h_\alpha))$. Then $B(e_\alpha, f_\alpha) = 2/\alpha(h_\alpha)$. Hence, by part (c),

$$[e_\alpha, f_\alpha] = \frac{2\, h_\alpha}{\alpha(h_\alpha)} = h^*_\alpha. \tag{8.10}$$

Note that $\alpha(h^*_\alpha) = 2$. From equation (8.10), we obtain the commutation relations

$$[h^*_\alpha, e_\alpha] = \alpha(h^*_\alpha)\, e_\alpha = 2\, e_\alpha \tag{8.11}$$

$$[h^*_\alpha, f_\alpha] = -\alpha(h^*_\alpha)\, f_\alpha = -2\, f_\alpha, \tag{8.12}$$

which show that the span of $(e_\alpha, f_\alpha, h^*_\alpha)$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, the explicit isomorphism mapping e_α to e, f_α to f and h^*_α to h.

□

An ordered triple (e', f', h') of nonzero elements of $\mathfrak{g}$ satisfying $[h', e'] = 2e'$, $[h', f'] = -2f'$, $[e', f'] = h'$ is called an $\mathfrak{sl}_2$-*triple*. Clearly, any $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is the basis of a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Assertion (f) above states that $(e_\alpha,\ f_\alpha,\ h_\alpha^*)$ is an $\mathfrak{sl}_2$-triple.

The appearance in semisimple algebras of 'copies' of $\mathfrak{sl}(2, \mathbb{C})$, the representation theory of which was completely elucidated in Theorem 7.3.12, has important consequences in the study of the geometry of roots, starting with the following integrality result.

Lemma 8.2.12. *If α and β are roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, then*

$$2\,\frac{B(\beta, \alpha)}{B(\alpha, \alpha)} \in \mathbb{Z}.$$

Proof. Using the notation of Theorem 8.2.11, Part (f), the triple $(e_\alpha, f_\alpha, h_\alpha^*)$ generates a three-dimensional simple Lie subalgebra $\mathfrak{g}^{(\alpha)}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Now $\mathfrak{g}^{(\alpha)}$ acts on $\mathfrak{g}$ via the adjoint representation, so by Weyl's Theorem (Theorem 7.3.7), $\mathfrak{g}$ is a direct sum of irreducible $\mathfrak{g}^{(\alpha)}$-invariant subspaces. By Theorem 7.3.12, each irreducible subspace has a basis consisting of eigenvectors of $\operatorname{ad} h_\alpha^*$, whose corresponding eigenvalues are all integers. Thus, $\mathfrak{g}$ has a basis consisting of eigenvectors of $\operatorname{ad} h_\alpha^*$ with integer eigenvalues. This implies in particular that *any* eigenvalue of $\operatorname{ad} h_\alpha^*$ is an integer.

Suppose that $\beta \in \Delta$. Then any nonzero vector $x_\beta \in \mathfrak{g}_\beta$ is an eigenvector of $\operatorname{ad} h_\alpha^*$, with eigenvalue

$$\beta(h_\alpha^*) = 2\,\frac{\beta(h_\alpha)}{\alpha(h_\alpha)} = 2\,\frac{B(\beta, \alpha)}{B(\alpha, \alpha)}.$$

□

Theorem 8.2.13. *Let Δ be the set of roots of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. Suppose that $\alpha \in \Delta$. Then* $\dim \mathfrak{g}_\alpha = 1$. *The only multiples of α which are roots are $\pm\alpha$.*

Proof. Suppose that $c\alpha$ is a root, for some $c \in \mathbb{C}$. Then by the integrality condition of Lemma 8.2.12,

$$2\,\frac{B(c\alpha, \alpha)}{B(\alpha, \alpha)} = 2c \in \mathbb{Z},$$

and hence $c = m/2$ for some nonzero integer m. Thus any multiple of a root which is also a root must be a half integer multiple of that root. If $|c| > 1$, then, since $\alpha = (1/c)\beta$, and $|1/c| < 1$, this forces $1/c = \pm 1/2$; that is $c = \pm 2$. If $|c| < 1$, then we must have $c = \pm 1/2$.

We conclude, therefore, that the only multiples of α, besides $\pm\alpha$, which can be roots are $\pm\alpha/2$ and $\pm 2\alpha$. It is clear that $\alpha/2$ and 2α *cannot both* be roots, since $\alpha/2 = (1/4)\, 2\alpha$.

Let us now show that the root space $\mathfrak{g}_\alpha$ is one-dimensional and that 2α cannot be a root. For this, let $\mathfrak{s}$ be the subspace of $\mathfrak{g}$ given by

$$\mathfrak{s} = \mathbb{C} f_\alpha \oplus \mathbb{C} h_\alpha^* \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

It is straightforward to show, using Theorem 8.2.11, Part (d), and the fact that $k\alpha$ is not a root, for $k > 2$, that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$, and that $h_\alpha^* \in [\mathfrak{s}, \mathfrak{s}]$. Let $\operatorname{ad}_{\mathfrak{s}}$ denote the adjoint representation on $\mathfrak{s}$. Since $h_\alpha^* \in \mathfrak{s}'$, we see that $\operatorname{tr}(\operatorname{ad}_{\mathfrak{s}} h_\alpha^*) = 0$. But since $\alpha(h_\alpha^*) = 2$,

$$\begin{aligned}\operatorname{tr}(\operatorname{ad}_{\mathfrak{s}} h_\alpha^*) &= -\alpha(h_\alpha^*) + 0 + (\dim \mathfrak{g}_\alpha)\, \alpha(h_\alpha^*) + (\dim \mathfrak{g}_{2\alpha})\, 2\alpha(h_\alpha^*) \\ &= -2 + 0 + 2 \dim \mathfrak{g}_\alpha + 4 \dim \mathfrak{g}_{2\alpha}\end{aligned}$$

Since $\dim \mathfrak{g}_\alpha \geq 1$, we therefore conclude that $\dim \mathfrak{g}_\alpha = 1$ and $\dim \mathfrak{g}_{2\alpha} = 0$. In particular, $2\alpha \notin \Delta$.

From this, we also see that $\alpha/2$ cannot be a root. If it were, then by the argument above, $2(\alpha/2) = \alpha$ cannot be a root, a contradiction.

This completes the proof of Theorem 8.2.13. □

The following theorem is a direct consequence of the representation theorem of $\mathfrak{sl}(2, \mathbb{C})$ (Theorem 7.3.12) and is a basis of the theory of Weyl groups and general Coxeter groups.

Theorem 8.2.14. *Suppose that α and β are roots of a complex semsimple Lie algebra $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$, with $\alpha \neq \pm\beta$. Let q be the largest integer j such that $\beta + j\alpha$ is a root, and let p be the smallest integer k such that $\beta + k\alpha$ is a root. Then:*

(i) For every j between p and q, $\beta + j\alpha$ is a root.

(ii) $\beta(h_\alpha^*) = 2B(\beta, \alpha)/B(\alpha, \alpha) = -(p + q)$.

(iii) $\beta - \beta(h_\alpha^*)\alpha$ is a root.

(iv) $[\mathfrak{g}_\beta, \mathfrak{g}_\alpha] = \mathfrak{g}_{\beta+\alpha}$ if $\beta + \alpha$ is a root.

Remark: By Part (i), the set of roots $\{\beta + j\alpha \,|\, p \leq j \leq q\}$ forms a connected string, which we call the *α-string through β*.

Proof of Theorem 8.2.14: Note that $q \geq 0$ and $p \leq 0$.

As in the proof of the Lemma 8.2.12, we let $\mathfrak{g}^{(\alpha)}$ be the three-dimensional simple Lie subalgebra of $\mathfrak{g}$ with basis $(e_\alpha, f_\alpha, h^*_\alpha)$, which, by Theorem 8.2.11, Part (f), is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Now let V be the subspace of $\mathfrak{g}$ given by

$$V = \bigoplus_{j=p}^{q} \mathfrak{g}_{\beta+j\alpha}, \tag{8.13}$$

where of course $\mathfrak{g}_{\beta+j\alpha} = \{0\}$ if $\beta+j\alpha$ is not a root. It is clear that V is invariant under $\operatorname{ad} e_\alpha$, $\operatorname{ad} f_\alpha$, and $\operatorname{ad} h^*_\alpha$, and so V is $\operatorname{ad} \mathfrak{g}^{(\alpha)}$-invariant. By Weyl's theorem (Theorem 7.3.7), V therefore decomposes into a direct sum of irreducible $\operatorname{ad} \mathfrak{g}^{(\alpha)}$-invariant subspaces:

$$V = V_1 \oplus \cdots \oplus V_l \tag{8.14}$$

By Theorem 7.3.12, each invariant irreducible subspace V_i further decomposes into a direct sum of one-dimensional eigenspaces of $\operatorname{ad} h^*_\alpha$, with the eigenvalues of $\operatorname{ad} h^*_\alpha$ on each V_i being integers all having the same parity (i.e., all odd or all even), and corresponding to a symmetric string $\{k, k-2, \dots, -(k-2), -k\}$ about 0. Thus V has a basis consisting of eigenvectors of $\operatorname{ad} h^*_\alpha$, a fact already evident from (8.13).

From equation (8.13), the eigenvalues of $\operatorname{ad} h^*_\alpha$ on V are of the form $(\beta + j\alpha)(h^*_\alpha) = \beta(h^*_\alpha) + 2j$, so they all have the same parity; moreover, the corresponding eigenspaces $\mathfrak{g}_{\beta+j\alpha}$ are all at most one-dimensional. It follows that there is just one irreducible component in the sum (8.14); that is, V is already irreducible.

We now apply Theorem 7.3.12 to V. The largest eigenvalue of $\operatorname{ad} h^*_\alpha$ on V is $(\beta+q\alpha)(h^*_\alpha) = \beta(h^*_\alpha)+2q$, the smallest eigenvalue is $(\beta+p\alpha)(h^*_\alpha) = \beta(h^*_\alpha)+2p$, and they are negatives of each other: $\beta(h^*_\alpha)+2p = -(\beta(h^*_\alpha)+2q)$. This proves (ii). Moreover, the integers in the string $\beta(h^*_\alpha)+2j$, $p \le j \le q$ are all eigenvalues of $\operatorname{ad} h^*_\alpha$. Hence $\mathfrak{g}_{\beta+j\alpha} \neq \{0\}$ for all $p \le j \le q$. This proves (i).

Since $q \ge 0$ and $p \le 0$, we have $p \le p+q \le q$, so by (ii), $\beta - \beta(h^*_\alpha)\alpha = \beta + (p+q)\alpha \in \Delta$. This proves (iii). Finally, if $\beta+\alpha \in \Delta$, then $q \ge 1$, so by Theorem 7.3.12, Part (6), $\operatorname{ad} e_\alpha\,(\mathfrak{g}_\beta) \neq \{0\}$. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq \{0\}$. This proves (iv).

Theorem 8.2.15. *Let α and β be roots, and let $\beta + k\alpha$ ($p \le k \le q$) be the α-string through β. Let x_α, $x_{-\alpha}$, and x_β be any vectors in $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{g}_\beta$, respectively. Then*

$$[x_{-\alpha}, [x_\alpha, x_\beta]] = \frac{q(1-p)\,\alpha(h_\alpha)}{2}\, B(x_\alpha, x_{-\alpha})\, x_\beta. \tag{8.15}$$

Proof. Note that if any of x_α, $x_{-\alpha}$, or x_β is 0, then both sides in equation (8.15) equal 0. Hence we may assume that these vectors are all nonzero.

Let $(e_\alpha,\, f_\alpha,\, h^*_\alpha)$ be the $\mathfrak{sl}_2$-triple whose existence is guaranteed by Theorem 8.2.11, Part (f), and let $\mathfrak{g}^{(\alpha)}$ denote the subalgebra, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, it generates in $\mathfrak{g}$.

Since $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are one-dimensional, there exist nonzero complex constants c_α and $c_{-\alpha}$ such that $x_\alpha = c_\alpha\, e_\alpha$ and $x_{-\alpha} = c_{-\alpha}\, f_\alpha$. By Theorem 8.2.11, Part (c), we can obtain a relation between c_α and $c_{-\alpha}$:

$$\begin{aligned} B(x_\alpha, x_{-\alpha})h_\alpha &= [x_\alpha, x_{-\alpha}] \\ &= c_\alpha\, c_{-\alpha}\, [e_\alpha, f_\alpha] \\ &= c_\alpha\, c_{-\alpha}\, h^*_\alpha \\ &= 2\, \frac{c_\alpha\, c_{-\alpha}}{\alpha(h_\alpha)}\, h_\alpha. \end{aligned}$$

Hence

$$c_\alpha\, c_{-\alpha} = \frac{\alpha(h_\alpha)}{2}\, B(x_\alpha, x_{-\alpha}).$$

By the proof of Theorem 8.2.14, $\mathfrak{g}^{(\alpha)}$ acts irreducibly on the subspace $\oplus_{k=p}^{q} \mathfrak{g}_{\beta+j\alpha}$ of $\mathfrak{g}$ via the adjoint representation, with highest weight $n = (\beta + q\alpha)(h^*_\alpha) = -(p+q) + 2q = q - p$.

Let v_0 be any nonzero element of $\mathfrak{g}_{\beta+q\alpha}$. Then by the representation theorem for $\mathfrak{sl}(2,\mathbb{C})$ (Theorem 7.3.12), we know that $(\operatorname{ad} f_\alpha)^q\, (v_0) = c\, x_\beta$, for some nonzero constant c. Now replace v_0 by $(1/c)\, v_0$, so that we may assume that $(\operatorname{ad} f_\alpha)^q\, (v_0) = x_\beta$.

According to the notation of Theorem 7.3.12, $(\operatorname{ad} f_\alpha)^q\, (v_0)$ is the vector v_q. From Part 6 of that theorem, we obtain the relation

$$\operatorname{ad} e_\alpha\, (v_q) = q(n - q + 1)v_{q-1}$$

and so

$$\operatorname{ad} f_\alpha \circ \operatorname{ad} e_\alpha\, (v_q) = q(n - q + 1)\, v_q.$$

Thus,

$$[f_\alpha, [e_\alpha, x_\beta]] = q(1 - p)\, x_\beta.$$

Multiplying both sides above by $c_\alpha\, c_{-\alpha}$, we obtain

$$[x_{-\alpha}, [x_\alpha, x_\beta]] = q(1 - p)\, \frac{\alpha(h_\alpha)}{2}\, B(x_\alpha, x_{-\alpha})\, x_\beta,$$

as desired. □

For the rest of this section, we assume that Δ is the set of roots of a complex semisimple Lie algebra $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$. Our next objective is to show that the Killing form B is positive definite on the real linear span of the root vectors h_α, for all $\alpha \in \Delta$. This linear span, which we denote by $h_{\mathbb{R}}$, will thus be a real inner product space.

Theorem 8.2.16. *Let $B = (\alpha_1, \dots, \alpha_r)$ be any basis of $\mathfrak{h}^*$ consisting of roots. (This is possible by Theorem 8.2.11, Part (a).) Then any root β is a linear combination of the α_j, with rational coefficients.*

Proof. Let $\beta \in \Delta$. Then we can certainly write $\beta = \sum_{j=1}^r c_j\,\alpha_j$, where $c_j \in \mathbb{C}$ for all j. Hence, for any i, we obtain

$$B(\beta, \alpha_i) = \sum_{j=1}^r c_j\, B(\alpha_j, \alpha_i).$$

By Theorem 8.2.11, Part (e), $B(\alpha_i, \alpha_i) \neq 0$. Hence

$$\frac{2\,B(\beta, \alpha_i)}{B(\alpha_i, \alpha_i)} = \sum_{j=1}^r c_j\, \frac{2\,B(\alpha_j, \alpha_i)}{B(\alpha_i, \alpha_i)} \tag{8.16}$$

For any i and j in $\{1, \dots, r\}$, let

$$n_i = 2\frac{B(\beta, \alpha_i)}{B(\alpha_i, \alpha_i)} \qquad \text{and} \qquad A_{ji} = 2\frac{B(\alpha_j, \alpha_i)}{B(\alpha_i, \alpha_i)}.$$

Then by Theorem 8.2.14, Part (ii), or by Lemma 8.2.12, all the n_i and A_{ji} are integers. Now the linear system (8.16) corresponds to the matrix equation

$$\begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix} = \begin{pmatrix} n_1 & \cdots & n_r \end{pmatrix}. \tag{8.17}$$

The coefficient matrix (A_{ji}) of the above matrix equation is equals the product

$$\begin{pmatrix} B(\alpha_1, \alpha_1) & \cdots & B(\alpha_1, \alpha_r) \\ \vdots & \ddots & \vdots \\ B(\alpha_r, \alpha_1) & \cdots & B(\alpha_r, \alpha_r) \end{pmatrix} \begin{pmatrix} 2/B(\alpha_1, \alpha_1) & & 0 \\ & \ddots & \\ 0 & & 2/B(\alpha_r, \alpha_r) \end{pmatrix}$$

Since B is nondegenerate on $\mathfrak{h}^*$, the matrix on the left above is nonsingular by Theorem 1.10.5; the matrix on the right is clearly nonsingular since it is a diagonal matrix with nonzero entries on the diagonal. This shows that (A_{ji}) is a nonsingular matrix with integer entries. Its inverse $(A_{ji})^{-1}$ is therefore a matrix with *rational* entries. Now we can solve for the coefficients c_j in the system (8.17) by multiplying both sides on the right by$(A_{ji})^{-1}$:

$$\begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix} = \begin{pmatrix} n_1 & \cdots & n_r \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix}^{-1}.$$

This shows that $c_j \in \mathbb{Q}$, for all j. □

Lemma 8.2.17. *Let α and β be roots. Then $B(\alpha, \beta) \in \mathbb{Q}$.*

Proof. For any $h \in \mathfrak{h}$, the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\sigma\in\Delta} \mathfrak{g}_\sigma$ is a decomposition of $\mathfrak{g}$ into eigenspaces of $\operatorname{ad} h$ with eigenvalues 0, of multiplicity $\dim \mathfrak{h}$, and $\sigma(h)$ (for all $\sigma \in \Delta$), of multiplicity 1.

For α and β in Δ, let h_α and h_β be the corresponding vectors in $\mathfrak{h}$, in accordance with Definition 8.2.10. Then

$$\begin{aligned} B(\alpha, \beta) &= B(h_\alpha, h_\beta) \\ &= \operatorname{tr}\,(\operatorname{ad} h_\alpha \circ \operatorname{ad} h_\beta) \\ &= \sum_{\sigma\in\Delta} \sigma(h_\alpha)\sigma(h_\beta) \\ &= \sum_{\sigma\in\Delta} B(\sigma, \alpha)\, B(\sigma, \beta). \end{aligned} \tag{8.18}$$

By Theorem 8.2.11 Part (e), $B(\alpha, \alpha) \neq 0$ and $B(\beta, \beta) \neq 0$. Hence we can divide both sides of (8.18) by these to get

$$\frac{4\, B(\alpha, \beta)}{B(\alpha, \alpha)\, B(\beta, \beta)} = \sum_{\sigma\in\Delta} \frac{2\, B(\sigma, \alpha)}{B(\alpha, \alpha)} \cdot \frac{2\, B(\sigma, \beta)}{B(\beta, \beta)} \tag{8.19}$$

Now by Theorem 8.2.14, Part (ii), the terms in the sum in the right hand side of (8.19) are integers. Hence the left hand side of (8.19) is an integer.

We want to prove that $B(\alpha, \beta)$ is rational. If $B(\alpha, \beta) = 0$, there is nothing to prove, so let us assume that $B(\alpha, \beta) \neq 0$. Then, again from Theorem 8.2.14, Part (ii), the fraction

$$\frac{4\, B(\alpha, \beta)^2}{B(\alpha, \alpha)\, B(\beta, \beta)} = \frac{2\, B(\beta, \alpha)}{B(\alpha, \alpha)} \cdot \frac{2\, B(\alpha, \beta)}{B(\beta, \beta)}$$

is an integer. Dividing this by the (nonzero) integer representing the left hand side of (8.19), we see that

$$B(\alpha, \beta) \;=\; \frac{\dfrac{4\, B(\alpha, \beta)^2}{B(\alpha, \alpha)\, B(\beta, \beta)}}{\dfrac{4\, B(\alpha, \beta)}{B(\alpha, \alpha)\, B(\beta, \beta)}}$$

is rational. □

Theorem 8.2.18. *Let $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha\in\Delta} \mathbb{R}\, h_\alpha$, the real vector space spanned by the vectors h_α ($\alpha \in \Delta$). Then the Killing form B is positive definite on $\mathfrak{h}_{\mathbb{R}}$.*

Proof. Certainly, Lemma 8.2.17 implies that $B(h_1, h_2) \in \mathbb{R}$ for all h_1, $h_2 \in \mathfrak{h}_{\mathbb{R}}$. Thus B is a real-valued symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}$. We want to prove that $B(h, h) > 0$ for all nonzero vectors h in $\mathfrak{h}_{\mathbb{R}}$.

For any $h \in \mathfrak{h}$, the root space decomposition (8.9) shows that the eigenvalues of the semisimple linear operator $\operatorname{ad} h$ on $\mathfrak{g}$ are 0 and $\alpha(h)$, for all $\alpha \in \Delta$. Thus, if $\alpha(h) = 0$ for all roots α, it would follow that $\operatorname{ad} h = 0$, and so $h = 0$. Consequently, if $h \neq 0$ in $\mathfrak{h}$, then $\alpha(h) \neq 0$ for some root α.

Now suppose that $h \in \mathfrak{h}_{\mathbb{R}}$. Then $h = \sum_{\alpha\in\Delta} c_\alpha h_\alpha$ for some scalars $c_\alpha \in \mathbb{R}$. For any root σ, the scalar $\sigma(h) = \sum_{\alpha\in\Delta} c_\alpha \sigma(h_\alpha) = \sum_{\alpha\in\Delta} c_\alpha B(\sigma, \alpha)$ is real, by Lemma 8.2.17. Therefore, by the root space decomposition,

$$\begin{aligned} B(h,h) &= \operatorname{tr}\,(\operatorname{ad} h \circ \operatorname{ad} h) \\ &= \sum_{\sigma\in\Delta} \sigma(h)^2 > 0 \end{aligned}$$

whenever $h \neq 0$ in $\mathfrak{h}_{\mathbb{R}}$. □

We conclude from this theorem that $\mathfrak{h}_{\mathbb{R}}$ is a real inner product space, with inner product given by the Killing form B. The map $h \mapsto B(h, \cdot)$ (which takes h_α to α, for all $\alpha \in \Delta$) identifies $\mathfrak{h}_{\mathbb{R}}$ with its real dual $\mathfrak{h}_{\mathbb{R}}^*$.

Corollary 8.2.19. *Suppose α and β are roots such that $B(\beta, \alpha) < 0$. Then $\beta + \alpha$ is a root. If $B(\beta, \alpha) > 0$, then $\beta - \alpha$ is a root.*

Proof. Suppose that $B(\beta, \alpha) < 0$. Now by Theorem 8.2.14, Part (ii), $2\,B(\beta, \alpha)/B(\alpha, \alpha) = -(p + q)$. Since $B(\alpha, \alpha) = B(h_\alpha, h_\alpha) > 0$, this implies that $q > 0$. Hence by Part (i), $\beta + \alpha$ is a root. If $B(\beta, \alpha) > 0$, then $p < 0$, so $\beta - \alpha$ is a root. □

8.3 Uniqueness of the Root Pattern

In this section our objective is to prove that if two complex semisimple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ have the same root pattern, then they are isomorphic. More precisely, we will prove the following theorem.

Theorem 8.3.1. *Suppose that $\mathfrak{g}$ and $\mathfrak{g}'$ are complex semisimple Lie algebras with Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. Let Δ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and let Δ' be the set of roots of $\mathfrak{g}'$ relative to $\mathfrak{h}'$. If $\varphi : \mathfrak{h} \to \mathfrak{h}'$ is a linear bijection such that ${}^t\varphi(\Delta') = \Delta$, then φ extends to an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.*

(Note: Here $\mathfrak{g}'$ *does not* refer to the derived algebra of $\mathfrak{g}$. It is just some other complex semisimple Lie algebra.)

Recall that ${}^t\varphi$ is the linear map from the dual space $(\mathfrak{h}')^*$ into the dual space $\mathfrak{h}^*$ given by ${}^t\varphi(\lambda) = \lambda \circ \varphi$, for all $\lambda \in (\mathfrak{h}')^*$. Since $\Delta \subset \mathfrak{h}^*$ and $\Delta' \subset (\mathfrak{h}')^*$, the requirement ${}^t\varphi(\Delta') = \Delta$ in the theorem above makes sense.

We will follow the arguments in [**?**], Chapter 3, §5.

For now, we focus on a complex semisimple Lie algebra $\mathfrak{g}$, its Cartan subalgebra $\mathfrak{h}$, and the set of roots Δ of $\mathfrak{g}$ relative to $\mathfrak{h}$. For each $\alpha \in \Delta$, we choose fix a vector $e_\alpha \in \mathfrak{g}_\alpha$. We can choose the e_α to have the property that $B(e_\alpha, e_{-\alpha}) = 1$ for all $\alpha \in \Delta$. Then by Theorem 8.2.11 Part (c), $[e_\alpha, e_{-\alpha}] = h_\alpha$, for all $\alpha \in \Delta$.

Now let S be a subset of Δ. The *hull* $\overline{S}$ of S is the set of all roots of the form $\pm\alpha$, $\pm(\alpha + \beta)$, for all α, $\beta \in S$. Thus $\overline{S} \subset \Delta$.

Suppose that γ and δ are in the hull $\overline{S}$ with $\gamma + \delta \neq 0$, and suppose that either $\gamma + \delta \in \overline{S}$ or $\gamma + \delta \notin \Delta$. We define the complex scalar $N_{\gamma,\delta}$ as follows. If $\gamma + \delta \in \overline{S}$, then $[e_\gamma, e_\delta] \in \mathfrak{g}_{\gamma+\delta}$, so we have $[e_\gamma, e_\delta] = N_{\gamma,\delta} e_{\gamma+\delta}$, where $N_{\gamma,\delta}$ is uniquely determined. If $\gamma + \delta \notin \Delta$, put $N_{\gamma,\delta} = 0$.

Thus $N_{\gamma,\delta}$ is defined for $\gamma, \delta \in \overline{S}$ when and only when:

1. $\gamma + \delta \neq 0$, and
2. $\gamma + \delta \in \overline{S}$, or $\gamma + \delta \notin \Delta$.

Clearly, $N_{\delta,\gamma} = -N_{\gamma,\delta}$.

Proposition 8.3.2. *Suppose that α, β, $\gamma \in \overline{S}$ such that $\alpha + \beta + \gamma = 0$. Then $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$.*

Proof. Note that the constants $N_{\alpha,\beta}$, $N_{\beta,\gamma}$, and $N_{\gamma,\alpha}$ are all defined. By the Jacobi identity

$$\begin{aligned} [e_\alpha, [e_\beta, e_\gamma]] &= -[e_\beta, [e_\gamma, e_\alpha]] - [e_\gamma, [e_\alpha, e_\beta]] \\ \Longrightarrow N_{\beta,\gamma}\, h_\alpha &= -N_{\gamma,\alpha}\, h_\beta \,-\, N_{\alpha,\beta}\, h_\gamma \end{aligned}$$

But $h_\alpha = -h_\beta - h_\gamma$, and so

$$-N_{\beta,\gamma}\, h_\beta \,-\, N_{\beta,\gamma}\, h_\gamma \,=\, -N_{\gamma,\alpha}\, h_\beta \,-\, N_{\alpha,\beta}\, h_\gamma$$

Now β and γ are linearly independent (otherwise $\beta = \pm\gamma$), and so h_β and h_γ are linearly independent. The last equation above thus establishes the proposition. □

Corollary 8.3.3. *Suppose that α, $\beta \in \overline{S}$ such that $N_{\alpha,\beta}$ exists. Then*

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -\frac{q(1-p)}{2}\, \alpha(h_\alpha), \tag{8.20}$$

where, as usual, $\beta + j\alpha$ $(p \leq j \leq q)$ is the α-string through β.

Proof. If $\alpha + \beta \notin \Delta$, then $q = 0$, so both sides of (8.20) are 0. Thus we can assume that $\alpha + \beta \in \overline{S}$. Now by Theorem 8.2.15,

$$\begin{aligned}\frac{q(1-p)}{2}\,\alpha(h_\alpha)\,e_\beta &= [e_{-\alpha}, [e_\alpha, e_\beta]] \\ &= N_{-\alpha,\alpha+\beta}\,N_{\alpha,\beta}\,e_\beta,\end{aligned}$$

and so $N_{-\alpha,\alpha+\beta}N_{\alpha,\beta} = q(1-p)\alpha(h_\alpha)/2$. Applying Proposition 8.3.2 to the triple $-\alpha$, $\alpha+\beta$, $-\beta$, we obtain $N_{-\alpha,\alpha+\beta} = N_{\alpha+\beta,-\beta} = N_{-\beta,-\alpha}$. Then using $N_{-\beta,-\alpha} = -N_{-\alpha,-\beta}$, we obtain the conclusion. □

Proposition 8.3.4. *Suppose that α, β, γ, $\delta \in S$ such that $\alpha+\beta+\gamma+\delta = 0$, and that no two of them sum to 0. Then $N_{\alpha,\beta}N_{\gamma,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} + N_{\alpha,\delta}N_{\beta,\gamma} = 0$.*

Proof. Since α, β, γ, and δ are all in S and since no two of them sum to 0, all the N's above are defined.

We start with the Jacobi identity on e_β, e_γ, and e_δ:

$$[e_\beta, [e_\gamma, e_\delta]] + [e_\gamma, [e_\delta, e_\beta]] + [e_\delta, [e_\beta, e_\gamma]] = 0. \tag{8.21}$$

Since $\beta+\gamma+\delta = -\alpha$, each of the three brackets on the left hand side above belongs to $\mathfrak{g}_{-\alpha}$. Consider the first bracket $[e_\beta, [e_\gamma, e_\delta]]$. If $\gamma+\delta$ is a root, then $\gamma+\delta \in \overline{S}$ and Theorem 8.2.14 implies that $N_{\gamma,\delta} \neq 0$. Hence

$$[e_\beta, [e_\gamma, e_\delta]] = N_{\beta,\gamma+\delta}\,N_{\gamma,\delta}\,e_{-\alpha}.$$

Now since $\beta+(\gamma+\delta)+\alpha = 0$, Proposition 8.3.2 implies that $N_{\beta,\gamma+\delta} = N_{\gamma+\delta,\alpha} = N_{\alpha,\beta}$. Thus

$$[e_\beta, [e_\gamma, e_\delta]] = N_{\alpha,\beta}\,N_{\gamma,\delta}\,e_{-\alpha}.$$

If $\gamma+\delta$ is not a root, then $[e_\gamma, e_\delta] = 0$ and $N_{\gamma,\delta} = 0$ by definition. Thus the equality above still holds, trivially.

The same reasoning can be applied to the other two brackets in (8.21), and we obtain

$$[e_\gamma, [e_\delta, e_\beta]] = N_{\delta,\beta}\,N_{\alpha,\gamma}\,e_{-\alpha}, \quad [e_\delta, [e_\beta, e_\gamma]] = N_{\alpha,\delta}\,N_{\beta,\gamma}\,e_{-\alpha}.$$

Since $N_{\gamma,\alpha} = -N_{\alpha,\gamma}$ and $N_{\beta,\delta} = -N_{\delta,\beta}$, equation (8.21) now becomes

$$(N_{\alpha,\beta}N_{\gamma,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} + N_{\alpha,\delta}N_{\beta,\gamma})\,e_{-\alpha} = 0,$$

proving the proposition.

□

For convenience, we will introduce what is called a lexicographic order to the real dual space $\mathfrak{h}_\mathbb{R}^*$, and thus to $\mathfrak{h}_\mathbb{R}$. An *ordered vector space* is a pair $(V, >)$

consisting of real vector space V and a total ordering $>$ on V which is preserved under vector addition and multiplication by positive scalars. Thus we require that for any u, v, and w in V and any positive scalar c,

$$u > v \implies u + w > v + w \text{ and } cu > cv.$$

Any vector $v > 0$ is called a *positive vector*; if $v < 0$, we call v a *negative vector*.

Let V be any real vector space. For a fixed basis $B = (v_1, \dots, v_n)$ of V, we can turn V into an ordered vector space by introducing the *lexicographic ordering relative to* B, defined as follows: for any nonzero vector $v \in V$, let us write $v = a_1 v_1 + \cdots + a_n v_n$. Let j be the smallest integer such that $a_j \neq 0$. By definition, $v > 0$ iff $a_j > 0$. Then, if v and w are any vectors in V, we define $v > w$ iff $v - w > 0$. It is straightforward to prove that $>$ is a total ordering on V which turns V into an ordered vector space.

Exercise 8.3.5. Suppose that $(V, >)$ is an ordered vector space. Prove that there is a basis B of V such that the total order $>$ is the lexicographic order relative to B.

We are now ready to prove Theorem 8.3.1.

Proof of Theorem 8.3.1: Let us first fix notation. For each $\alpha \in \Delta$, let α' be the unique element of Δ' such that ${}^t\varphi(\alpha') = \alpha$. Thus, in particular, $\alpha'(\varphi(h)) = \alpha(h)$ for all $h \in \mathfrak{h}$.

Let B and B' denote the Killing forms on $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. The first thing we'll do is to prove that φ is an *isometry* with respect to B and B'. More precisely, we will prove that

$$B'(\varphi(h_1), \varphi(h_2)) = B(h_1, h_2) \tag{8.22}$$

for all $h_1,\, h_2 \in \mathfrak{h}$.

In fact, if $\alpha,\, \beta \in \Delta$, then

$$\begin{aligned} B'(\varphi(h_\alpha), \varphi(h_\beta)) &= \operatorname{tr}(\operatorname{ad}\varphi(h_\alpha) \circ \operatorname{ad}\varphi(h_\beta)) \\ &= \sum_{\gamma' \in \Delta'} \gamma'(\varphi(h_\alpha))\, \gamma'(\varphi(h_\beta)) \\ &= \sum_{\gamma \in \Delta} \gamma(h_\alpha)\, \gamma(h_\beta) \\ &= B(h_\alpha, h_\beta) \end{aligned}$$

Since every vector in $\mathfrak{h}$ is a $\mathbb{C}$-linear combination of the h_α, equation (8.22) follows by bilinearity.

For all $\alpha \in \Delta$ and $h \in \mathfrak{h}$, we then see that $B'(\varphi(h_\alpha), \varphi(h)) = B(h_\alpha, h) = \alpha(h) = \alpha'(\varphi(h)) = B'(h_{\alpha'}, \varphi(h))$. Since B' is nondegenerate on $\mathfrak{h}'$, this implies that

$\varphi(h_\alpha) = h_{\alpha'}$. From this, we obtain $B(h_\alpha, h_\beta) = B'(h_{\alpha'}, h_{\beta'})$ for all α, $\beta \in \Delta$; moreover, $\varphi(\mathfrak{h}_\mathbb{R}) = \mathfrak{h}'_\mathbb{R}$.

The real dual space $\mathfrak{h}^*_\mathbb{R}$ is the $\mathbb{R}$-linear span $\sum_{\alpha\in\Delta} \mathbb{R}\alpha$, and likewise $(\mathfrak{h}'_\mathbb{R})^*$ is the $\mathbb{R}$-linear span of Δ'. Fix a basis E of $(\mathfrak{h}'_\mathbb{R})^*$, and let $>$ be the lexicographic order on $(\mathfrak{h}'_\mathbb{R})^*$ with respect to E. We also let $>$ be the lexicographic order on $\mathfrak{h}^*_\mathbb{R}$ with respect to its basis ${}^t\varphi(E)$. These orders are obviously compatible in the sense that $\lambda' > \mu'$ iff ${}^t\varphi(\lambda') > {}^t\varphi(\mu')$ for all λ', $\mu' \in (\mathfrak{h}'_\mathbb{R})^*$. In particular, $\alpha > \beta$ iff $\alpha' > \beta'$ for all α, $\beta \in \Delta$.

Let us apply the discussion prior to Proposition 8.3.2 to $S = \Delta$. Thus for all $\alpha \in \Delta$, we choose vectors $e_\alpha \in \mathfrak{g}_\alpha$ such that $B(e_\alpha, e_{-\alpha}) = 1$. Then, for each pair of roots α, $\beta \in \Delta$ such that $\alpha + \beta \neq 0$, the scalar $N_{\alpha,\beta} \in \mathbb{C}$ is defined by

$$[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$$

if $\alpha + \beta \in \Delta$, and $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Delta$. The $N_{\alpha,\beta}$ satisfy the conclusions of Proposition 8.3.2, Corollary 8.3.3, and Proposition 8.3.4.

For each $\alpha' \in \Delta'$, we claim that there exists a vector $e_{\alpha'} \in \mathfrak{g}'_{\alpha'}$ such that

$$B'(e_{\alpha'}, e_{-\alpha'}) = 1 \tag{8.23}$$
$$[e_{\alpha'}, e_{\beta'}] = N_{\alpha,\beta}\, e_{\alpha'+\beta'} \qquad (\text{if } \alpha' + \beta' \neq 0) \tag{8.24}$$

Assuming that the the vectors $e_{\alpha'}$ exist, we extend the linear map φ to all of $\mathfrak{g}$ by putting $\varphi(e_\alpha) = e_{\alpha'}$ for all α. Then, for all $h \in \mathfrak{h}$, we have

$$[\varphi(h), \varphi(e_\alpha)] = [\varphi(h), e_{\alpha'}] = \alpha'(\varphi(h))\, e_{\alpha'} = \alpha(h)\, e_{\alpha'} = \varphi[h, e_\alpha].$$

This relation and (8.24) then show that this extension of φ is our desired isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.

So everything hinges on proving the existence of the vectors $e_{\alpha'}$ satisfying equations (8.23) and (8.24).

We will do this by "induction" with respect to the lexicographic order $>$ on $\mathfrak{h}^*_\mathbb{R}$. For each positive root $\rho \in \Delta$, let Δ_ρ denote the set of all $\alpha \in \Delta$ such that $-\rho < \alpha < \rho$. For $\rho' \in \Delta'$, define $\Delta'_{\rho'}$ similarly.

Order the positive roots in Δ by $\rho_1 < \rho_2 < \cdots < \rho_m$, and let ρ_{m+1} be any vector in $\mathfrak{h}^*_\mathbb{R}$ larger than ρ_m. By induction on j, we will prove the following claim:

Claim: *For each $\alpha \in \Delta_{\rho_j}$, a vector $e_{\alpha'} \in \mathfrak{g}'_{\alpha'}$ can be chosen such that equation (8.23) holds whenever $\alpha \in \Delta_{\rho_j}$ and equation (8.24) holds whenever α, β, $\alpha+\beta \in \Delta_{\rho_j}$.*

Since $\Delta_{\rho_{m+1}} = \Delta$, the proof will be complete by the $(m + 1)$st induction step. Note that for all $j \leq m$, $\Delta_{\rho_{j+1}} = \Delta_{\rho_j} \cup \{\rho_j, -\rho_j\}$.

For $j = 1$, we have $\Delta_{\rho_1} = \emptyset$, so the claim certainly holds vacuously and there is nothing to prove. So assume that the claim holds for Δ_{ρ_j}. We wish to prove it for $\Delta_{\rho_{j+1}}$. For this, we just need to define $e_{\rho'_j}$ and $e_{-\rho'_j}$ in an appropriate manner so that the claim still holds for $\Delta_{\rho_{j+1}}$.

If there are *no* α, $\beta \in \Delta_{\rho_j}$ such that $\rho_j = \alpha + \beta$, then we can choose $e_{\rho'_j}$ to be any nonzero vector in $\mathfrak{g}'_{\rho'_j}$, and then let $e_{-\rho'_j}$ be the vector in $\mathfrak{g}'_{-\rho'_j}$ satisfying $B'(e_{\rho'_j}, e_{-\rho'_j}) = 1$. In this case, the claim then holds for $\Delta_{\rho_{j+1}}$. In fact, if α, β, $\alpha + \beta \in \Delta_{\rho_{j+1}}$, then the only cases that are not necessarily covered by the induction hypothesis occur when α or β equals $\pm\rho_j$. If $\alpha = \rho_j$, then $\alpha + \beta$ cannot be $\pm\rho_j$, and so we would have $\rho_j = (\alpha + \beta) + (-\beta)$, where both $\alpha + \beta$ and $-\beta$ are in Δ_{ρ_j}. This is impossible, since ρ_j is not the sum of elements in Δ_{ρ_j}. The other possibilities $\alpha = -\rho_j$, $\beta = \pm\rho_j$ also cannot occur. Thus, the only cases that occur are already covered by the induction hypothesis, and so the claim holds for $\Delta_{\rho_{j+1}}$.

So suppose that there are roots α, $\beta \in \Delta_{\rho_j}$ such that $\alpha + \beta = \rho_j$. Note that any such pair of roots must be positive: if, for instance, $\beta < 0$, then $\alpha = \rho_j - \beta > \rho_j$, contradicting $\alpha \in \Delta_{\rho_j}$.

Among all such pairs of roots, let α, β be the pair such that α is as small as possible, with respect to the ordering $>$. Define $e_{\rho'_j} \in \mathfrak{g}'_{\rho'_j}$ by the condition

$$[e_{\alpha'}, e_{\beta'}] = N_{\alpha,\beta}\, e_{\rho'_j}. \tag{8.25}$$

Then let $e_{-\rho'_j}$ be the vector in $\mathfrak{g}'_{-\rho'_j}$ such that $B'(e_{\rho'_j}, e_{-\rho'_j}) = 1$.

To prove that the claim holds for $\Delta_{\rho_{j+1}}$, we just need to verify that

$$[e_{\gamma'}, e_{\delta'}] = N_{\gamma,\delta}\, e_{\gamma'+\delta'} \tag{8.26}$$

whenever γ, δ, $\gamma + \delta \in \Delta_{\rho_{j+1}}$. For this, we'll need to consider several cases:

1. *γ, δ, and $\gamma+\delta$ belong to Δ_{ρ_j}.* By the induction hypothesis, equation (8.26) holds and there is nothing to prove.

2. $\gamma + \delta = \rho_j$. In this case, we can assume that $\{\gamma, \delta\} \neq \{\alpha, \beta\}$. Note that γ and δ are positive. Now $\alpha + \beta - \gamma - \delta = 0$, and no two of the roots α, β, $-\gamma$, $-\delta$ have sum 0. Thus by Proposition 8.3.4 (for $S = \Delta$), we have

$$N_{\alpha,\beta}\, N_{-\gamma,-\delta} = -N_{-\gamma,\alpha}\, N_{\beta,-\delta} - N_{\alpha,-\delta}\, N_{\beta,-\gamma} \tag{8.27}$$

   Moreover, by Corollary 8.3.3,

$$N_{\gamma,\delta}\, N_{-\gamma,-\delta} = -\frac{l(1-k)}{2}\, \gamma(h_\gamma), \tag{8.28}$$

   where $\delta + s\gamma$, $k \leq s \leq l$ is the γ-string through δ.

Now this time, let $S = \{\alpha, \beta, -\gamma, -\delta\}$. As in the beginning of this section, for $\mu, \nu \in \overline{S}$ such that $\mu + \nu \neq 0$, we define the scalar $M_{\mu,\nu}$ under the condition that either $\mu + \nu \in \overline{S}$ or $\mu + \nu \notin \Delta$ by:

$$\begin{aligned} [e_{\mu'}, e_{\nu'}] &= M_{\mu,\nu}\, e_{\mu'+\nu'} && \text{if } \mu + \nu \in \overline{S} \\ M_{\mu,\nu} &= 0 && \text{if } \mu + \nu \notin \Delta \end{aligned} \tag{8.29}$$

By construction, we already have $M_{\alpha,\beta} = N_{\alpha,\beta}$. In addition, by the induction hypothesis, we also have $M_{\mu,\nu} = N_{\mu,\nu}$ whenever μ, ν, and $\mu+\nu$ are in Δ_{ρ_j}.

Now by Proposition 8.3.4 applied to the scalars M, we have

$$M_{\alpha,\beta}\, M_{-\gamma,-\delta} = -M_{-\gamma,\alpha}\, M_{\beta,-\delta} - M_{\alpha,-\delta}\, M_{\beta,-\gamma}. \tag{8.30}$$

But then the all the terms on the right hand side above coincide with the corresponding terms on the right hand side of equation (8.27). Hence $M_{-\gamma,-\delta} = N_{-\gamma,-\delta}$. Also, by Corollary 8.3.3, we have

$$M_{\gamma,\delta}\, M_{-\gamma,-\delta} = -\frac{l'(1-k')}{2}\, \gamma'(h_{\gamma'}) \tag{8.31}$$

where $\delta' + s\gamma'$, $k' \leq s \leq l'$ is the γ'-string through δ'. But $\gamma'(h_{\gamma'}) = \gamma(h_\gamma)$, and by the hypothesis ${}^t\varphi(\Delta') = \Delta$, we have ${}^t\varphi(\delta' + s\gamma') = \delta + s\gamma$, for all s. Hence $k = k'$ and $l = l'$. It follows that the right hand side in (8.31) equals that in (8.28), whence $M_{\gamma,\delta}\, N_{-\gamma,-\delta} = N_{\gamma,\delta}\, N_{-\gamma,-\delta}$. It follows that $M_{\gamma,\delta} = N_{\gamma,\delta}$.

3. $\gamma + \delta = -\rho_j$. Then $-\gamma - \delta = \rho_j$. By Case 2, we have $[e_{-\gamma'}, e_{-\delta'}] = N_{-\gamma,-\delta}\, e_{\rho_j'}$. Let $S = \{\gamma, \delta, -\rho_j\}$. Then define the scalars $M_{\mu,\nu}$ for $\overline{S}$ in a manner analogous to (8.29). Then

$$M_{\gamma,\delta}\, M_{-\gamma,-\delta} = -\frac{l(1-k)}{2}\, \gamma(h_\gamma) = N_{\gamma,\delta}\, N_{-\gamma,-\delta},$$

where $\delta + s\gamma$, $k \leq s \leq l$ is the γ-string through δ. Since $M_{-\gamma,-\delta} = N_{-\gamma,-\delta}$, it follows from the above that $M_{\gamma,\delta} = N_{\gamma,\delta}$. Thus, $[e_{\gamma'}, e_{\delta'}] = N_{\gamma,\delta}\, e_{\gamma'+\delta'}$.

4. *One of γ or δ is $\pm\rho_j$.* Suppose, for instance, that $\gamma = -\rho_j$. Then by $-\rho_j \leq -\rho_j + \delta \leq \rho_j$, we obtain $\delta > 0$; since $\delta \leq \rho_j$, we then conclude that $-\rho_j + \delta \leq 0$; since $-\rho_j + \delta$ is a root, we must have $-\rho_j + \delta < 0$. Thus, in fact $\delta < \rho_j$.

From this we obtain that $\rho_j = \delta + (\rho_j - \delta)$, where both $\rho_j - \delta$ and δ lie in Δ_{ρ_j}. Now let $S = \{\delta, \rho_j - \delta, -\rho_j\}$, and define scalars $M_{\mu,\nu}$ for $\mu, \nu \in \overline{S}$ as in (8.29). From Case 2, we have $M_{\delta,\rho_j-\delta} = N_{\delta,\rho_j-\delta}$. Then by Proposition 8.3.2, we have

$$M_{\delta,\rho_j-\delta} = M_{\rho_j-\delta,-\rho_j} = M_{-\rho_j,\delta}.$$

Since we also have

$$N_{\delta,\rho_j-\delta} = N_{\rho_j-\delta,-\rho_j} = N_{-\rho_j,\delta},$$

we conclude that $M_{-\rho_j,\delta} = N_{-\rho_j,\delta}$. The cases $\gamma = \rho_j$, $\delta = \pm\rho_j$ are treated in a similar fashion.

This completes the induction step and concludes the proof of Theorem 8.3.1. □