Chapter 7

Semisimple Lie Algebras: Basic Structure and Representations

7.1 The Basic Structure of a Semisimple Lie Algebra

The rest of the text is essentially going to be devoted to the structure theory of semisimple and simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$. We start off with an important consequence of Cartan's criterion for semisimplicity.

We say that a Lie algebra $\mathfrak{g}$ is a *direct sum of ideals* if there exist ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$. Note that if $i \neq j$, then $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_i \cap \mathfrak{a}_j = \{0\}$.

Theorem 7.1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is a direct sum of simple ideals*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k. \tag{7.1}$$

Any simple ideal of $\mathfrak{g}$ is one of the ideals $\mathfrak{g}_i$. Any ideal of $\mathfrak{g}$ is a direct sum of some of the $\mathfrak{g}_i$'s.

Proof. If $\mathfrak{g}$ is already simple, then we're done. So assume that $\mathfrak{g}$ is not simple. Then $\mathfrak{g}$ has ideals $\neq \{0\}$ and $\neq \mathfrak{g}$. Let $\mathfrak{g}_1$ be a non-zero ideal of $\mathfrak{g}$ of *minimal* dimension.

The subspace $\mathfrak{g}_1^\perp = \{x \in \mathfrak{g} \,|\, B(x, y) = 0 \text{ for all } y \in \mathfrak{g}_1\}$ is an ideal of $\mathfrak{g}$. In fact, if $x \in \mathfrak{g}_1^\perp$ and $y \in \mathfrak{g}$, then for any $z \in \mathfrak{g}_1$, we have by Lemma 6.4.1,

$$B([x,y],z) = B(x, \underbrace{[y,z]}_{\text{in } \mathfrak{g}_1}) = 0, \tag{7.2}$$

so $[x,y] \in \mathfrak{g}_1^\perp$.

Next we prove that $[\mathfrak{g}_1, \mathfrak{g}_1^\perp] = \{0\}$. For this, let $u \in \mathfrak{g}_1$ and $v \in \mathfrak{g}_1^\perp$. Then for any $w \in \mathfrak{g}$, we have

$$B([u,v],w) = B(u, \underbrace{[v,w]}_{\text{in } \mathfrak{g}_1^\perp}) = 0.$$

Since B is nondegenerate, we conclude that $[u,v] = 0$.

It follows that

$$[\mathfrak{g}_1 \cap \mathfrak{g}_1^\perp, \mathfrak{g}_1 \cap \mathfrak{g}_1^\perp] \subset [\mathfrak{g}_1, \mathfrak{g}_1^\perp] = \{0\},$$

and hence $\mathfrak{g}_1 \cap \mathfrak{g}_1^\perp$ is an abelian ideal of $\mathfrak{g}$. But since $\mathfrak{g}$ is semisimple, this means that $\mathfrak{g}_1 \cap \mathfrak{g}_1^\perp = \{0\}$.

In addition, since B is nondegenerate, equation (1.34) says that $\dim \mathfrak{g} = \dim \mathfrak{g}_1 + \dim \mathfrak{g}_1^\perp$. Together with our observation that $\mathfrak{g}_1 \cap \mathfrak{g}_1^\perp = \{0\}$, we see that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp. \tag{7.3}$$

Now, by Proposition 6.4.3, the Killing form on the ideal $\mathfrak{g}_1^\perp$ is the restriction of B to $\mathfrak{g}_1^\perp \times \mathfrak{g}_1^\perp$. But B is nondegenerate on $\mathfrak{g}_1^\perp$. In fact, if $x \in \mathfrak{g}_1^\perp$ satisfies $B(x, \mathfrak{g}_1^\perp) = \{0\}$, then we also have $B(x, \mathfrak{g}) = B(x, \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp) = \{0\}$, so $x = 0$. By Cartan's criterion for semisimplicity, we conclude that $\mathfrak{g}_1^\perp$ is semisimple.

Next we observe that $\mathfrak{g}_1$ is a simple ideal of $\mathfrak{g}$. In fact, by the decomposition (7.3), any ideal of $\mathfrak{g}_1$ is also an ideal of $\mathfrak{g}$. Then, by the minimality of $\dim \mathfrak{g}_1$, such an ideal is either $\{0\}$ or $\mathfrak{g}_1$.

We now apply the procedure above to the semisimple ideal $\mathfrak{g}_1^\perp$ in place of $\mathfrak{g}$ to produce ideals $\mathfrak{g}_2$ and $\mathfrak{g}''$ of $\mathfrak{g}_1^\perp$, with $\mathfrak{g}_2$ simple and $\mathfrak{g}''$ semisimple, such that

$$\mathfrak{g}_1^\perp = \mathfrak{g}_2 \oplus \mathfrak{g}''.$$

Then by (7.3),

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}''.$$

The decomposition above shows that both $\mathfrak{g}_2$ and $\mathfrak{g}''$ are ideals of $\mathfrak{g}$. We then apply the same procedure to $\mathfrak{g}''$, etc., to produce the direct sum (7.1) of simple ideals of $\mathfrak{g}$.

Now suppose that $\mathfrak{m}$ is an ideal of $\mathfrak{g}$. Let $I = \{i \,|\, \mathfrak{g}_i \subset \mathfrak{m}\}$ and let $J = \{1, \ldots, k\} \setminus I$. We claim that $\mathfrak{m} = \bigoplus_{i \in I} \mathfrak{g}_i$. Certainly, $\mathfrak{m} \supset \bigoplus_{i \in I} \mathfrak{g}_i$. Now suppose that $x \in \mathfrak{m} \setminus \bigoplus_{i \in I} \mathfrak{g}_i$. We decompose x according to the direct sum (7.1) to obtain

$$x = x_I + x_J,$$

where

$$x_I \in \mathfrak{g}_I := \bigoplus_{i \in I} \mathfrak{g}_i \qquad \text{and} \qquad x_J \in \mathfrak{g}_J := \bigoplus_{j \in J} \mathfrak{g}_j.$$

Since $x \notin \mathfrak{g}_I$, we have $x_J \neq 0$. But $x_I \in \mathfrak{g}_I \subset \mathfrak{m}$, so we see that $x_J \in \mathfrak{m}$. If $[x_J, \mathfrak{g}_j] = \{0\}$ for all $j \in J$, then $[x_J, \mathfrak{g}] = \{0\}$, so x_J lies in the center $\mathfrak{c}$ of $\mathfrak{g}$, so $x_J = 0$, contrary to $x_J \neq 0$. Thus $[x_J, \mathfrak{g}_j] \neq \{0\}$ for some $j \in J$. For this j, we conclude that $[\mathfrak{m}, \mathfrak{g}_j] \neq \{0\}$, and hence $\mathfrak{m} \cap \mathfrak{g}_j \neq \{0\}$. Since $\mathfrak{g}_j$ is simple, it follows that $\mathfrak{g}_j \subset \mathfrak{m}$, so $j \in I$, a contradiction. This shows that $\mathfrak{m} = \bigoplus_{i \in I} \mathfrak{g}_i$.

Finally, if $\mathfrak{m}$ is a simple ideal of $\mathfrak{g}$, then there is only one summand in $\mathfrak{m} = \bigoplus_{i \in I} \mathfrak{g}_i$, so $\mathfrak{m} = \mathfrak{g}_i$ for some i. □

Corollary 7.1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. By Theorem 7.1.1, $\mathfrak{g}$ is a direct sum of simple ideals: $\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i$. Since each $\mathfrak{g}_i$ is simple, we have $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, and since the sum is direct, we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$ for $i \neq j$. Hence

$$\begin{aligned}
[\mathfrak{g}, \mathfrak{g}] &= \left[\bigoplus_{i=1}^{k} \mathfrak{g}_i, \bigoplus_{j=1}^{k} \mathfrak{g}_j \right] \\
&= \sum_{i,j} [\mathfrak{g}_i, \mathfrak{g}_j] \\
&= \sum_{i} \mathfrak{g}_i \\
&= \bigoplus_{i=1}^{k} \mathfrak{g}_i \\
&= \mathfrak{g}.
\end{aligned}$$

□

Exercise 7.1.3. Prove the converse to Theorem 7.1.1: If $\mathfrak{g}$ is a direct sum of simple ideals, then $\mathfrak{g}$ is semisimple.

Theorem 7.1.1 and Exercise 7.1.3 show that the study of semisimple Lie algebras over $\mathbb{F}$ essentially reduces to the study of *simple* Lie algebras over $\mathbb{F}$.

Corollary 7.1.4. *If $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{F}$, then so are all ideals of $\mathfrak{g}$ and all homomorphic images of $\mathfrak{g}$.*

Proof. By Theorem 7.1.1, $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_i$, and any ideal of $\mathfrak{g}$ is a direct sum of some of the $\mathfrak{g}_i$. By Exercise 7.1.3, any such ideal must be semisimple.

If $\mathfrak{m}$ is a homomorphic image of $\mathfrak{g}$, then $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Now $\mathfrak{a}$ is a direct sum $\mathfrak{a} = \bigoplus_{i \in I} \mathfrak{g}_i$, for some subset $I \subset \{1, \dots, k\}$. Put $J = \{1, \dots, n\} \setminus I$. Then $\mathfrak{m} \cong \bigoplus_{j \in J} \mathfrak{g}_j$, a semisimple Lie algebra by Exercise 7.1.3. □

Theorem 7.1.5. *Let $\mathfrak{g}$ be a semsimple Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is complete, that is,* $\operatorname{ad} \mathfrak{g} = Der\, \mathfrak{g}$.

We conclude this section with an important theorem, whose proof we shall omit. (See [Jac79], Chapter III, §9.)

Theorem 7.1.6. *(The Levi Decomposition) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and let $\mathcal{R}_s$ be its solvable radical. Then $\mathfrak{g}$ is a direct sum of ideals $\mathfrak{g} = \mathcal{R}_s \oplus \mathcal{I}$, where the ideal $\mathcal{I}$ is semisimple.*

The semisimple ideal $\mathcal{I}$, which is not unique, is called a *Levi factor* of $\mathfrak{g}$. If $\mathcal{I}_1$ is another Levi factor of $\mathfrak{g}$, then there is an automorphism φ of $\mathfrak{g}$ such that $\varphi(\mathcal{I}) = \mathcal{I}_1$.

7.2 Simple Lie Algebras over $\mathbb{R}$

In this section we obtain a general characterization of simple Lie algebras over $\mathbb{R}$. It turns out that there are essentially two types, depending on their complexifications.

Theorem 7.2.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$. Then $\mathfrak{g}$ is exactly one of the following two types:*

1. *A real form of a simple Lie algebra over $\mathbb{C}$*
2. *A simple Lie algebra over $\mathbb{C}$, considered as a real Lie algebra.*

$\mathfrak{g}$ is of the second type if and only if its complexification $\mathfrak{g}^c$ is the direct sum of two simple ideals, both isomorphic (as real Lie algebras) to $\mathfrak{g}$.

Proof. We can assume that $\mathfrak{g} \neq \{0\}$. The Lie algebra $\mathfrak{g}$ is, of course, semisimple because of Exercise 7.1.3. Then by Lemma 6.4.7, the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ is semisimple. By Theorem 7.1.1, $\mathfrak{g}^c$ is the direct sum of simple ideals

$$\mathfrak{g}^c = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m. \tag{7.4}$$

Now let σ denote the conjugation of $\mathfrak{g}^c$ with respect to its real form $\mathfrak{g}$. The image $\sigma(\mathfrak{g}_1)$ is closed with respect to multiplication by complex scalars, since if $z \in \mathbb{C}$ and $v \in \mathfrak{g}_1$, then $z\,\sigma(v) = \sigma(\overline{z}\,v) \in \sigma(\mathfrak{g}_1)$. Thus $\sigma(\mathfrak{g}_1)$ is a complex vector subspace of $\mathfrak{g}^c$. It is also an ideal of $\mathfrak{g}^c$ since

$$\begin{aligned}[\sigma(\mathfrak{g}_1), \mathfrak{g}^c] &= [\sigma(\mathfrak{g}^c), \sigma(\mathfrak{g}_1)] \\ &= \sigma\left([\mathfrak{g}^c, \mathfrak{g}_1]\right) \\ &= \sigma(\mathfrak{g}_1).\end{aligned}$$

Finally, $\sigma(\mathfrak{g}_1)$ is a *simple* ideal of $\mathfrak{g}^c$: if $\mathfrak{a}$ is any ideal of $\sigma(\mathfrak{g}_1)$, then $\sigma(\mathfrak{a})$ is an ideal of $\mathfrak{g}_1$, so $\sigma(\mathfrak{a}) = \mathfrak{g}_1$ or $\sigma(\mathfrak{a}) = \{0\}$. Since σ is bijective, this forces $\mathfrak{a} = \sigma(\mathfrak{g}_1)$ or $\mathfrak{a} = \{0\}$.

Thus $\sigma(\mathfrak{g}_1)$ must be one of the ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$. Suppose first that $\sigma(\mathfrak{g}_1) = \mathfrak{g}_1$. Then $\mathfrak{g}_1$ is σ-invariant. Let $\mathfrak{a} = \{v \in \mathfrak{g}_1 \,|\, \sigma(v) = v\}$. Clearly, $\mathfrak{a} = \mathfrak{g} \cap \mathfrak{g}_1$, so $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Each $x \in \mathfrak{g}_1$ can be written as

$$x = \frac{x + \sigma(x)}{2} + i\,\frac{i(\sigma(x) - x)}{2}$$

Both $(x + \sigma(x)/2$ and $i(\sigma(x) - x)/2$ belong to $\mathfrak{a}$, which shows that

$$\mathfrak{g}_1 = \mathfrak{a} \oplus i\mathfrak{a}$$

as real vector spaces. We conclude that $\mathfrak{a}$ is a non-zero ideal of $\mathfrak{g}$, whence $\mathfrak{a} = \mathfrak{g}$. Thus $\mathfrak{g}_1 = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{g}^c$, and so $\mathfrak{g}$ is a real form of the complex simple Lie algebra $\mathfrak{g}_1$.

Suppose next that $\sigma(\mathfrak{g}_1) = \mathfrak{g}_j$ for some $j \geq 2$. Let $\mathfrak{h} = \mathfrak{g}_1 \oplus \sigma(\mathfrak{g}_1)$. $\mathfrak{h}$ is then a non-zero σ-invariant ideal of $\mathfrak{g}^c$. The same reasoning as that in the preceding paragraph then shows that $\mathfrak{h} = \mathfrak{g}^c$, and so

$$\mathfrak{g}^c = \mathfrak{g}_1 \oplus \sigma(\mathfrak{g}_1).$$

Thus $\mathfrak{g}^c$ is the direct sum of two simple (complex) ideals. The map

$$x \mapsto x + \sigma(x)$$

is then easily shown to be a real Lie algebra isomorphism from $\mathfrak{g}_1$ onto $\mathfrak{g}$. (See the exercise below.) Thus $\mathfrak{g}$ is isomorphic to a complex simple Lie algebra, considered as a real Lie algebra. □

Exercise 7.2.2. In the last part of the proof of Theorem 7.2.1, show that $x \mapsto x + \sigma(x)$ is an real Lie algebra isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}$.

The complete classification of complex simple Lie algebras was carried out by Cartan and Killing in the early part of the twentieth century. This also classifies the real simple Lie algebras of type (2) above. The classification of the real forms of complex simple Lie algebras is a much harder problem, and is related to the classification of symmetric spaces. This was also completed by Cartan in the 1930's.

7.3 Basic Representation Theory

In this section, we introduce some of the basic terminology and results of the representation theory of Lie algebras, such as the complete reducibility of $\mathfrak{g}$-modules when $\mathfrak{g}$ is semisimple, Schur's Lemma, and the representation theory of $\mathfrak{sl}(2,\mathbb{C})$.

Definition 7.3.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. A vector space V over $\mathbb{F}$ is called a $\mathfrak{g}$-*module* if there is a representation π of $\mathfrak{g}$ on V.

Recall that we also say that $\mathfrak{g}$ *acts on* V.

Definition 7.3.2. Let $\pi_1 : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\pi_2 : \mathfrak{g} \to \mathfrak{gl}(W)$ be representations of the Lie algebra $\mathfrak{g}$. A linear map $T : V \to W$ is said to *intertwine* π_1 and π_2 if $\pi_2(x) \circ T = T \circ \pi_1(x)$, for all $x \in \mathfrak{g}$. We also say that T is a $\mathfrak{g}$-*equivariant* linear map from the $\mathfrak{g}$-module V to the $\mathfrak{g}$-module W.

Thus T intertwines the representations π_1 and π_2 if, for all $x \in \mathfrak{g}$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\pi_1(x)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\pi_2(x)]{} & W \end{array}$$

If T is a linear isomorphism, we call T a $\mathfrak{g}$-*module isomorphism.* In this case, it is clear that T^{-1} is also a $\mathfrak{g}$-module isomorphism from W onto V.

Definition 7.3.3. Let V be a $\mathfrak{g}$-module, via the representation π. A subspace U of V is called a $\mathfrak{g}$-*submodule* if W is invariant under all operators $\pi(x)$, for all $x \in \mathfrak{g}$. Thus the map $\pi_U : \mathfrak{g} \to \mathfrak{gl}(U)$ given by $\pi_U(x) = \pi(x)|_U$ is a representation of $\mathfrak{g}$ on U.

If U is a $\mathfrak{g}$-submodule of V, we also say that U is a $\mathfrak{g}$-*invariant subspace* of V. Note that the sum and the intersection of $\mathfrak{g}$-invariant subspaces is a $\mathfrak{g}$-invariant subspace. In addition, if U is a $\mathfrak{g}$-invariant subspace, then the quotient space V/U is a $\mathfrak{g}$-module via the *quotient representation* π' given by

$$\pi'(x)\,(v+U) = \pi(x)\,(v) + U \tag{7.5}$$

for all $x \in \mathfrak{g}$ and all $v \in V$. (The relation $\pi'[x,y] = [\pi'(x), \pi'(y)]$ follows immediately from $\pi[x,y] = [\pi(x), \pi(y)]$.) We call V/U a *quotient module.*

Definition 7.3.4. A representation π of $\mathfrak{g}$ on a vector space V is said to be *irreducible* if V has no $\mathfrak{g}$-submodules other than $\{0\}$ and V. We also say that V is an *irreducible* $\mathfrak{g}$-module.

One more definition:

Definition 7.3.5. A representation π of $\mathfrak{g}$ on a vector space V is said to be *completely reducible* if, for any $\mathfrak{g}$-invariant subspace U of V, there exists a $\mathfrak{g}$-invariant subspace W of V such that $V = U \oplus W$.

Example 7.3.6. A Lie algebra $\mathfrak{g}$ acts on itself via the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. A subspace $\mathfrak{a} \subset \mathfrak{g}$ is $\mathfrak{g}$-invariant if and only if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. The adjoint representation is completely reducible if and only if, for any ideal $\mathfrak{a}$ of $\mathfrak{g}$, there is another ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

If $\mathfrak{g}$ is semisimple, then ad is completely reducible. In fact, by Theorem 7.1.1, $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$. If $\mathfrak{a}$ is any ideal of $\mathfrak{g}$, then $\mathfrak{a} = \oplus_{i \in I} \mathfrak{g}_i$, for some subset I of $\{1, \dots, n\}$. Put $J = \{1, \dots, n\} \setminus I$, and let $\mathfrak{b} = \oplus_{j \in J} \mathfrak{g}_j$. Then $\mathfrak{b}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

In the above example, there's nothing special about the representation ad. What's important is that $\mathfrak{g}$ is semisimple, as the following theorem shows:

Theorem 7.3.7. *(H. Weyl) Let $\mathfrak{g}$ be a semismple Lie algebra over $\mathbb{F}$. Then any representation π of $\mathfrak{g}$ is completely reducible.*

The proof, which we omit, can be found in several places, such as [**?**], §6. For convenience, we include a proof in Appendix **??**

Here is an equivalent characterization of complete reducibility.

Theorem 7.3.8. *Let V be a vector space over $\mathbb{F}$ and let π be a representation of a Lie algebra $\mathfrak{g}$ on V. Then π is completely reducible if and only if V is a direct sum of irreducible $\mathfrak{g}$-modules:*

$$V = V_1 \oplus \cdots \oplus V_m \tag{7.6}$$

Proof. Suppose that π is completely reducible. If V is already irreducible, then there is nothing to prove. Otherwise, choose a $\mathfrak{g}$-invariant subspace V_1 of V, of *minimum* positive dimension. Clearly, V_1 is an irreducible $\mathfrak{g}$-module. Since π is completely reducible, V_1 has a complementary $\mathfrak{g}$-invariant subspace W, so

$$V = V_1 \oplus W \tag{7.7}$$

If W is irreducible, then let $V_2 = W$, and we are done. If it isn't, there exists an $\mathfrak{g}$-invariant subspace V_2 of W, of minimum positive dimension. Then V_2 is irreducible. Moreover, V_2 has a complementary $\mathfrak{g}$-invariant subspace W' in V:

$$V = V_2 \oplus W' \tag{7.8}$$

We now claim that

$$W = V_2 \oplus (W' \cap W). \tag{7.9}$$

In fact, by equation (7.8) any $w \in W$ can be written as $w = v_2 + w'$, where $v_2 \in V_2$ and $w' \in W'$. Since $v_2 \in W$, it follows that $w' \in W$, so $w' \in W \cap W'$. Thus $W = V_2 + (W \cap W')$. The sum is direct, since $V_2 \cap (W \cap W') \subset V_2 \cap W' = \{0\}$. This proves (7.9), and so by (7.7),

$$V = V_1 \oplus V_2 \oplus (W \cap W') \tag{7.10}$$

The subspace $W'' = W \cap W'$ is an intersection of $\mathfrak{g}$-invariant subspaces, which is $\mathfrak{g}$-invariant. Thus V is the direct sum of $\mathfrak{g}$-invariant subspaces

$$V = V_1 \oplus V_2 \oplus W'' \tag{7.11}$$

If W'' is irreducible, put $V_3 = W''$ and we're done. If not, let V_3 be a $\mathfrak{g}$-invariant subspace of W'' of minimum positive dimension. Then V_3 is irreducible, and, just as we obtained the decomposition (7.11), we can write V as a direct sum of $\mathfrak{g}$-submodules

$$V = V_1 \oplus V_2 \oplus V_3 \oplus W^{(3)}. \tag{7.12}$$

If we continue this procedure, we will eventually reach the decomposition (7.6) above, since $\dim V$ is finite.

Conversely, suppose that π is a representation of $\mathfrak{g}$ on V, and that V is a direct sum (7.6) of irreducible $\mathfrak{g}$-modules. We want to prove that π is completely reducible. Let U be a $\mathfrak{g}$-invariant subspace of V, with $U \neq \{0\}$ and $U \neq V$.

Since $U \neq V$, there is a subspace V_{i_1} among the irreducible subspaces in (7.6) such that $V_{i_1} \not\subseteq U$. Thus $V_{i_1} \cap U$ is a proper $\mathfrak{g}$-invariant subspace of V_{i_1}; since V_{i_1} is irreducible, we conclude that $V_{i_1} \cap U = \{0\}$. Put $U_2 = U \oplus V_{i_1}$. If $U_2 = V$, then we can take V_{i_1} as our complementary $\mathfrak{g}$-invariant subspace. if $U_2 \neq V$, there is another irreducible subspace V_{i_2} in (7.6) such that $V_{i_2} \not\subseteq U_2$. Then $V_{i_2} \cap U_2 = \{0\}$, so we can let

$$U_3 = U_2 \oplus V_{i_2} = U \oplus V_{i_1} \oplus V_{i_2}.$$

If $U_3 = V$, then we can take our complementary $\mathfrak{g}$-invariant subspace to be $W = V_{i_1} \oplus V_{i_2}$. If $U_3 \neq V$, then there is a subspace V_{i_3} among the irreducible subspaces in (7.6) such that $V_{i_3} \not\subseteq U_3$, and so forth. Since V is finite-dimensional, this procedure ends after a finite number of steps, and we have

$$V = U \oplus V_{i_1} \oplus \cdots \oplus V_{i_r}.$$

The subspace $W = V_{i_1} \oplus \cdots \oplus V_{i_r}$ is then our $\mathfrak{g}$-invariant complementary subspace to U. □

Some authors define complete reducibility by means of the decomposition (7.6). In general, neither this decomposition nor the complementary $\mathfrak{g}$-invariant subspace in the definition of complete reducibility is unique.

Example 7.3.9. Consider the representation π of $\mathfrak{gl}(2,\mathbb{C})$ on itself via matrix multiplication:

$$\pi(X)\,(Y) = X\,Y$$

It is easy to see that π is indeed a representation, and that the representation space $\mathfrak{gl}(2,\mathbb{C})$ decomposes into the direct sum of irreducible subspaces:

$$\mathfrak{gl}(2,\mathbb{C}) = \left\{ \left(\begin{array}{cc} z_1 & 0 \\ z_2 & 0 \end{array} \right) \middle| z_1, z_2 \in \mathbb{C} \right\} \oplus \left\{ \left(\begin{array}{cc} 0 & z_3 \\ 0 & z_4 \end{array} \right) \middle| z_3, z_4 \in \mathbb{C} \right\}$$

$\mathfrak{gl}(2,\mathbb{C})$ *also* decomposes into the following invariant irreducible subspaces

$$\mathfrak{gl}(2,\mathbb{C}) = \left\{ \left(\begin{array}{cc} z_1 & z_1 \\ z_2 & z_2 \end{array} \right) \middle| z_1, z_2 \in \mathbb{C} \right\} \oplus \left\{ \left(\begin{array}{cc} z_3 & -z_3 \\ z_4 & -z_4 \end{array} \right) \middle| z_3, z_4 \in \mathbb{C} \right\}$$

Theorem 7.3.10. *(Schur's Lemma) Let V be a vector space over $\mathbb{F}$, and let π be an irreducible representation of $\mathfrak{g}$ on V. If $T \in \mathcal{L}(V)$ commutes with $\pi(x)$, for all $x \in \mathfrak{g}$, then either $T = 0$ or T is invertible. If $\mathbb{F} = \mathbb{C}$, then T is just scalar multiplication: $T = \lambda I_V$.*

Proof. First we observe that both the kernel and the range of T are $\mathfrak{g}$-invariant subspaces of V. In fact, if $v \in \ker T$, then $T(\pi(x)\,v) = \pi(x)\,(T(v)) = 0$, so $\pi(x)\,v \in \ker T$ for all $x \in V$. Likewise, $\pi(x)\,(T(V)) = T(\pi(x)(V)) \subset T(V)$.

Since $\ker T$ is $\mathfrak{g}$-invariant and π is irreducible, we have either $\ker T = \{0\}$ or $\ker T = V$. In the former case, T is invertible, and in the latter case, $T = 0$.

Suppose now that $\mathbb{F} = \mathbb{C}$. Then our linear operator T has an eigenvalue λ, so $\ker(T - \lambda I_V) \neq \{0\}$. But the operator $T - \lambda I_V$ commutes with $\pi(x)$, for all $x \in \mathfrak{g}$. Thus, by the preceding paragraph, $T - \lambda I_V = 0$, and so $T = \lambda I_V$. □

Exercise 7.3.11. A symmetric bilinear form Q on a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is called $\mathfrak{g}$-*invariant* provided that

$$Q([x,y],z) = -Q(x,[y,z])$$

for all $x, y, z \in \mathfrak{g}$. For example, the Killing form B on $\mathfrak{g}$ is $\mathfrak{g}$-invariant.

(a) If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$, prove that any $\mathfrak{g}$-invariant symmetric bilinear form Q on $\mathfrak{g}$ is a constant multiple of the Killing form B.

(b) Let V be a vector space over $\mathbb{C}$. Suppose that $\mathfrak{g}$ is a simple Lie subalgebra of $\mathfrak{gl}(V)$. Prove that the Killing form B is a non-zero multiple of the trace form $Q(X,Y) = \operatorname{tr}(XY)$.

Hint for Part (a): Show that there exists a unique linear operator T on $\mathfrak{g}$ such that $Q(x,y) = B(Tx,y)$ for all $x, y \in \mathfrak{g}$. Then show that $T \circ \operatorname{ad} x = \operatorname{ad} x \circ T$ for all $x \in \mathfrak{g}$.

Now we turn to a simple but important topic: the representation theory of the three-dimensional simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. As we will see later on, this is the "glue" by which the structure of any complex simple Lie algebra is built upon. Now by Weyl's Theorem (Theorem 7.3.7), any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible, so to understand the (finite-dimensional) representation theory of this Lie algebra, it suffices, by Theorem 7.3.8, to study its irreducible representations.

Theorem 7.3.12. *(The Basic Representation Theorem for $\mathfrak{sl}(2,\mathbb{C})$.) Let $(e,\ f,\ h)$ be the standard basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, with*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let π be an irreducible representation of $\mathfrak{g}$ on a complex vector space V. Then there exists an eigenvector v_0 of $\pi(h)$, with eigenvalue λ, such that $\pi(e)\, v_0 = 0$. For each $j \in \mathbb{Z}^+$, let $v_j = (\pi(f))^j\, v_0$. Then the following properties hold:

1. *The eigenvalue λ is a non-negative integer n*
2. $v_{n+1} = 0$
3. $(v_0, v_1, \ldots, v_n)$ *is a basis of V*
4. $\pi(f)\, v_j = v_{j+1}$
5. $\pi(h)\, v_j = (n-2j)\, v_j$*, so each basis vector v_j is an eigenvector of $\pi(h)$*
6. $\pi(e)\, v_j = j(n-j+1)v_{j-1}$.

Remark: This theorem implies, among other things, that $\pi(h)$ is a semisimple linear operator on V with integer eigenvalues n, $(n-2)$, $\ldots, -(n-2)$, $-n$. On the other hand, both $\pi(e)$ and $\pi(f)$ are nilpotent operators, since their matrices with respect to the basis $(v_0, v_1, \ldots, v_n)$ of V are strictly upper and lower triangular, respectively.

Proof. Since V is a complex vector space, the linear operator $\pi(h)$ has a complex eigenvalue μ. Let v be an eigenvector of $\pi(h)$ corresponding to μ. We claim that $\pi(e)\, v$ belongs to the eigenspace of $\pi(h)$ corresponding to the eigenvalue $\mu + 2$. In fact,

$$\begin{aligned} \pi(h)\ (\pi(e)\, v) &= \pi(e)\ (\pi(h)\, v) + (\pi(h)\, \pi(e) - \pi(e)\, \pi(h))\ (v) \\ &= \mu\, \pi(e)\, v + [\pi(h), \pi(e)]\, v \\ &= \mu\, \pi(e)\, v + \pi[h,e]\,(v) \\ &= \mu\, \pi(e)\, v + 2\, \pi(e)\, v \\ &= (\mu+2)\ \pi(e)\, v. \end{aligned}$$

A similar argument then shows that $\pi(e)^2\,(v) = \pi(e)\;(\pi(e)\,v)$ belongs to the eigenspace of $\pi(h)$ corresponding to the eigenvalue $\mu + 4$. In general, $\pi(e)^s\,v$ belongs to the eigenspace of $\pi(h)$ corresponding to the eigenvalue $\mu + 2s$. Since $\pi(h)$ has only finitely many eigenvalues, we must have $\pi(e)^s\,v = 0$ for some $s \in \mathbb{N}$. Let s be the *smallest* positive integer such that $\pi(e)^s\,v \neq 0$ but $\pi(e)^{s+1}\,v = 0$. Put $v_0 = \pi(e)^s\,v$. Then $\pi(e)\,v_0 = 0$, and v_0 is an eigenvector of $\pi(h)$. Let λ $(= \mu + 2s)$ be its eigenvalue.

Now as prescribed in the statement of the theorem, for each $j \in \mathbb{N}$, we define $v_j = \pi(f)^j\,v_0$. This trivially gives conclusion (4). Let us now prove by induction that for each $j \in \mathbb{Z}^+$,

$$\pi(h)\,v_j = (\lambda - 2j)\,v_j. \tag{7.13}$$

If $j = 0$, the equation above is just $\pi(h)\,v_0 = \lambda\,v_0$, which is true by the hypothesis on v_0. Assume, then, that equation 7.13 is true for v_j. Then

$$\begin{aligned}
\pi(h)\,v_{j+1} &= \pi(h)\;(\pi(f)\,v_j)\\
&= \pi(f)\,(\pi(h)\,v_j) + [\pi(h), \pi(f)]\,(v_j)\\
&= (\lambda - 2j)\,\pi(f)(v_j) + \pi[h,f]\,(v_j) \qquad \text{(by induction hypothesis)}\\
&= (\lambda - 2j)\,v_{j+1} \;-\; 2\,\pi(f)\,(v_j)\\
&= (\lambda - 2j)\,v_{j+1} \;-\; 2\,v_{j+1}\\
&= (\lambda - 2(j+1))\,v_{j+1},
\end{aligned}$$

which proves that equation (7.13) is true for v_{j+1}.

Thus, if $v_j \neq 0$, it must be an eigenvector of $\pi(h)$ corresponding to the eigenvalue $\lambda - 2j$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent and V is finite-dimensional, we conclude that there must be a $j \in \mathbb{N}$ such that $v_j = 0$. Let n be the smallest non-negative integer such that $v_n \neq 0$ but $v_{n+1} = 0$.

We will now prove by induction that for all $j \geq 1$,

$$\pi(e)\,v_j = j(\lambda - j + 1)\,v_{j-1}. \tag{7.14}$$

Let us first verify equation (7.14) for $j = 1$. We have

$$\begin{aligned}
\pi(e)\,v_1 &= \pi(e)\;(\pi(f)\,v_0)\\
&= \pi(f)\;(\pi(e)\,v_0) \;+\; \pi[e,f]\,v_0\\
&= 0 + \pi(h)\,v_0\\
&= \lambda\,v_0,
\end{aligned}$$

which is precisely equation (7.14) for $j = 1$.

Next assume that equation 7.14 holds for v_j (with $j \geq 1$). Then

$$\begin{aligned}
\pi(e)\, v_{j+1} &= \pi(e)\ (\pi(f)\, v_j) \\
&= \pi(f)\ (\pi(e)\, v_j)\ +\ \pi[e,f]\, v_j \\
&= j(\lambda - j + 1)\, \pi(f)\, v_{j-1}\ +\ \pi(h)\, v_j \qquad \text{(by induction hypothesis)} \\
&= j(\lambda - j + 1)\, v_j\ +\ (\lambda - 2j)\, v_j \\
&= (j+1)(\lambda - j)\, v_j \\
&= (j+1)(\lambda - (j+1) + 1)\, v_j,
\end{aligned}$$

proving (7.14) for v_{j+1}.

If we now apply (7.14) to the vector $v_{n+1} = 0$, we get

$$0 = \pi(e)\, v_{n+1} = (n+1)(\lambda - n)\, v_n.$$

Since $v_n \neq 0$, we conclude that $(n+1)(\lambda - n) = 0$, and so $\lambda = n$. This proves conclusion (1). Plugging in $\lambda = n$ to equation (7.13), we obtain conclusion (5); and plugging this into equation (7.14) gives us conclusion (6).

It remains to prove conclusion (3), that $(v_0, \ldots, v_n)$ is a basis of V. These vectors are certainly linearly independent, since by (7.13), they are eigenvectors of $\pi(h)$ corresponding to distinct eigenvalues. From conclusions (4), (5), and (6), we also see that the $\mathbb{C}$-span of $(v_0, \ldots, v_n)$ is invariant under $\pi(e)$, $\pi(f)$, and $\pi(h)$. Since (e, f, h) is a basis of $\mathfrak{g}$, we see that this linear span is a $\mathfrak{g}$-invariant subspace. Since V is an irreducible $\mathfrak{g}$-module, we conclude that this span is all of V. This proves conclusion (3) and finishes the proof of Theorem 7.3.12. □

In Theorem 7.3.12, the non-negative integer n is called the *highest weight* of the representation π. The vector v_0 is called a *highest weight vector* of π; the vectors $v_0, \ldots, v_n$ are called *weight vectors*, and their eigenvalues n, $n-2$, $n-4$, $\ldots$, $-(n-2)$, $-n$ are called the *weights* of π.

Theorem 7.3.12 says that, up to $\mathfrak{g}$-module isomorphism, any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is uniquely determined by its highest weight. It also says that the representation space V has a basis $(v_0, \ldots, v_n)$ satisfying conditions (4)-(6) in the statement of the theorem.

Exercise 7.3.13. (*Converse of Theorem 7.3.12*) Fix a positive integer n, let V be a vector space over $\mathbb{C}$ with basis $(v_0, v_1, \ldots, v_n)$, and let π_n be the linear map from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{gl}(V)$ defined on the basis $(e,\ f,\ h)$ of $\mathfrak{sl}(2, \mathbb{C})$ by the relations (4)-(6) in Theorem 7.3.12. Prove that π_n is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$.

Actually, the representation π_n in Theorem 7.3.12 has an explicit realization. Namely, let V be the vector space of *homogeneous degree n polynomials* in two complex variables z and w, with complex coefficients. Thus the elements of V are polynomials of the form

$$p(z,w) = \alpha_n z^n + \alpha_{n-1} z^{n-1} w + \cdots + \alpha_1 z w^{n-1} + \alpha_0 w^n,$$

where $\alpha_n, \cdots, \alpha_0$ are complex numbers. The following $n+1$ degree n monomials

$$z^n,\ z^{n-1}w,\ \cdots,\ zw^{n-1},\ w^n$$

constitute a basis of V. For each matrix $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$, define the linear map $\pi(X)$ on V by

$$\left(\pi \begin{pmatrix} a & b \\ c & -a \end{pmatrix} p\right)(z,w) = (az + cw)\frac{\partial p}{\partial z} + (bz - aw)\frac{\partial p}{\partial w}$$

If $p(z,w)$ is homogeneous of degree n, it is clear that the right hand side above is also homogeneous of degree n. Since $\pi(X)$ is given by a linear differential operator, it is therefore clear that $\pi(X)$ is a linear operator on V.

Exercise 7.3.14. Prove that $X \mapsto \pi(X)$ is a Lie algebra homomorphism of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{gl}(V)$. For this you need to show that $X \mapsto \pi(X)$ is linear (straightforward), and that $\pi[X,Y] = [\pi(X), \pi(Y)]$, for all $X, Y \in \mathfrak{sl}(2,\mathbb{C})$. Show that this amounts to proving that for all polynomials p in two variables z and w,

$$\left[(a_1 z + c_1 w)\frac{\partial}{\partial z} + (b_1 z - a_1 w)\frac{\partial}{\partial w}, (a_2 z + c_2 w)\frac{\partial}{\partial z} + (b_2 z - a_2 w)\frac{\partial}{\partial w}\right](p)$$
$$= \big((b_1c_2 - c_1b_2)z + 2(c_1a_2 - a_1c_2)w\big)\frac{\partial p}{\partial z} + \big(2(a_1b_2 - b_1a_2)z - (b_1c_2 - c_1b_2)w\big)\frac{\partial p}{\partial w}$$

Exercise 7.3.15. Put $v_0 = z^n$ and for $j = 1, \ldots, n$, put $v_j = P(n,j) z^{n-j} w^j$, where $P(n,j) = n!/(n-j)!$. Then V has basis $(v_0, \ldots, v_n)$. Prove that these basis vectors satisfy the relations (4)-(6) in Theorem 7.3.12.

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