Chapter 6

Cartan's Criteria for Solvability and Semisimplicity

In this section we define an important symmetric bilinear form on a Lie algebra $\mathfrak{g}$ and derive conditions on this form which are necessary and sufficient for $\mathfrak{g}$ to be solvable, as well as conditions on the form which are necessary and sufficent for $\mathfrak{g}$ to be semisimple.

6.1 The Killing Form

For any elements x and y of $\mathfrak{g}$, the map $\operatorname{ad} x \circ \operatorname{ad} y$ is a linear operator on $\mathfrak{g}$, so we may consider its trace.

Definition 6.1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. The *Killing form* on $\mathfrak{g}$ is the map

$$\begin{aligned} B\colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{F} \\ (x, y) &\mapsto \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) \end{aligned} \tag{6.1}$$

Thus, $B(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y)$.

Note that $B(y, x) = \operatorname{tr}(\operatorname{ad} y \circ \operatorname{ad} x) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = B(x, y)$, so the Killing form B is symmetric.

Since the adjoint map ad and the trace are linear, it is also easy to see that

$B(x, y)$ is a bilinear form:

$$\begin{aligned} B(\alpha\, x_1 + \beta x_2, y) &= \operatorname{tr}(\operatorname{ad}(c\, x_1 + x_2) \circ \operatorname{ad} y) \\ &= \operatorname{tr}((\alpha\ \operatorname{ad} x_1 + \beta \operatorname{ad} x_2) \circ \operatorname{ad} y) \\ &= \alpha \operatorname{tr}(\operatorname{ad} x_1 \circ \operatorname{ad} y) + \beta \operatorname{tr}(\operatorname{ad} x_2 \circ \operatorname{ad} y) \\ &= \alpha\, B(x_1, y) + \beta B(x_2, y), \end{aligned}$$

for all x_1, x_2, $y \in \mathfrak{g}$, and all α, $\beta \in \mathbb{F}$. (The linearity of B in the second argument follows from its the above and the fact that B is symmetric.)

Exercise 6.1.2. (Graduate Exercise.) Suppose that $\mathfrak{g}$ is the Lie algebra of a compact Lie group G. Prove that B is negative semidefinite; i.e., $B(x, x) \leq 0$ for all $x \in \mathfrak{g}$. If $\mathfrak{c} = \{0\}$, show that B is negative definite. (*Hint:* There exists an inner product Q on $\mathfrak{g}$ invariant under $\operatorname{ad} G$: $B(\operatorname{Ad} g\,(x), \operatorname{Ad} g\,(y)) = B(x, y)$ for all x, $y \in \mathfrak{g}$ and all $g \in G$.)

Our objective in this section is to prove the following theorems.

Theorem 6.1.3. *(Cartan's Criterion for Solvability) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is solvable if and only if $B(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.*

Theorem 6.1.4. *(Cartan's Criterion for Semisimplicity) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is semisimple if and only if the Killing form B is nondegenerate.*

We will first prove Theorem 6.1.3 for complex Lie subalgebras of $\mathfrak{gl}(V)$, where V is a complex vector space. Then, in order to prove it for real Lie algebras, we will need to use the notion of complexification. Theorem 6.1.4 will then essentially be a corollary of Theorem 6.1.3.

Let us now develop the necessary machinery.

6.2 The Complexification of a Real Lie Algebra

A vector space V over $\mathbb{R}$ is said to have a *complex structure* if there is a $J \in \mathcal{L}(V)$ such that $J^2 = -I_V$. Note that, by definition, J is $\mathbb{R}$-linear, and that the condition $J^2 = -I_V$ means that it is invertible. Note also that J has no real eigenvalues, since its only possible eigenvalues are $\pm i$.

The operator J turns the vector space V into a complex vector space in which scalar multiplication by $z = \alpha + \beta i$ (with α, $\beta \in \mathbb{R}$) is given by

$$(\alpha + \beta i)\, v = \alpha\, v + \beta\, Jv, \tag{6.2}$$

for all $v \in V$. The routine verification that V is indeed a complex vector space will be left to the reader.

Example 6.2.1. For any real vector space U, let V be the external direct sum $V = U \oplus U$. Then the linear operator J on V given by $J\,(u_1, u_2) = (-u_2, u_1)$ is a complex structure on V. Since $(u_1, u_2) = (u_1, 0) + J(u_2, 0) = (u_1, 0) + i(u_2, 0)$, it is often convenient to identify U with the subspace $\{(u, 0)\,|\,u \in U\}$, and thus write the element (u_1, u_2) as $u_1 + Ju_2$. Since V now has a complex vector space structure, we call V the *complexification of* U, and denote it by U^c.

Note: Complexification can also be carried out using tensor products: $V = U \otimes_{\mathbb{R}} \mathbb{C}$, but we'll not go through this route.

Suppose that J is a complex structure on a real vector space V. Then V becomes a complex vector space, with scalar multiplcation given by (6.2) above. Since any $\mathbb{R}$-spanning set in V is also a $\mathbb{C}$-spanning set, it is obvious that V is finite-dimensional as a complex vector space. Now let $(u_1, \dots, u_n)$ be a $\mathbb{C}$-basis of V. Then $(u_1, \dots, u_n, Ju_1, \dots, Ju_n)$ is an $\mathbb{R}$-basis of V: in fact, any $v \in V$ can be written as a *unique* linear combination

$$v = \sum_{j=1}^{n} (\alpha_j + i\beta_j)\, u_j = \sum_{j=1}^{n} \alpha_j\, u_j \; + \sum_{j=1}^{n} \beta_j\, Ju_j \qquad\qquad (\alpha_j, \beta_j \in \mathbb{R})$$

Now if we let U be the real subspace $\mathbb{R}u_1 + \cdots + \mathbb{R}u_n$ of V, we see that $V = U \oplus JU$ (as a real vector space), and is thus easy to see that $V \cong U^c$. In particular, $\dim_{\mathbb{R}} V = 2n = 2 \dim_{\mathbb{C}} V$, so any real vector space with a complex structure is even-dimensional over $\mathbb{R}$. The subspace U is called a *real form* of V.

Of course, any complex vector space V is a real vector space equipped with a complex structure: $Jv = iv$, for all $v \in V$. In the future, we will nonetheless have occasion to complexify a *complex* vector space (considered as a real vector space) using the construction in Example 6.2.1.

So suppose that U is a complex vector space. Considering U as a real vector space, we can then complexify U in accordance with Example 6.2.1. Now the external direct sum $U^c = U \oplus U = U \times U$ is already a complex vector space, since each factor is a complex vector space. The complex structure J on $U \oplus U$ commutes with multiplication by i, since

$$J(i(u_1, u_2)) = J(iu_1, iu_2) = (-iu_2, iu_1) = i(-u_2, u_1) = iJ(u_1, u_2).$$

Thus J is a $\mathbb{C}$-linear map on $U \oplus U$. $U \oplus U$ decomposes into a direct sum of $\pm i$-eigenspaces of J:

$$(u_1, u_2) = \frac{1}{2}\,(u_1 + iu_2, u_2 - iu_1) + \frac{1}{2}\,(u_1 - iu_2, u_2 + iu_1),$$

so

$$U^c = \{(v, -iv)\,|\,v \in U\} \,\oplus\, \{(w, iw)\,|\,w \in U\}.$$

If U is a real form of complex vector space V, we define the *conjugation* τ_U *of* V *with respect to* U as follows: for any $v \in V$, we can write v uniquely as

$v = u_1 + iu_2$, where $u_1, u_2 \in U$; put $\tau_U(v) = u_1 - iu_2$. Then τ_U is an $\mathbb{R}$-linear map of V satisfying $\tau_U^2 = I_V$. It is easy to check that τ_U is conjugate-linear: $\tau_U(zv) = \overline{z}\,\tau_U(v)$, for all $v \in V$ and $z \in \mathbb{C}$.

Exercise 6.2.2. (i) Suppose that T is a $\mathbb{C}$-linear operator on a complex vector space V. Show that if T_R denotes T considered as an $\mathbb{R}$-linear operator on V, then $\operatorname{tr}(T_R) = 2\operatorname{Re}(\operatorname{tr}(T))$. (ii) Next suppose that T is an $\mathbb{R}$-linear operator on a real vector space U. Show that T has a unique natural extension T^c to a $\mathbb{C}$-linear map on U^c, and that $\operatorname{tr}(T_c) = \operatorname{tr}(T)$.

Now suppose that $\mathfrak{g}$ is a real Lie algebra equipped with a complex structure J. J is said to be *compatible* with the Lie bracket in $\mathfrak{g}$ if $[Jx, y] = J[x, y]$ for all $x, y \in \mathfrak{g}$. (Then, of course $[x, Jy] = J[x, y]$ for all x, y.) If $\mathfrak{g}$ is given the complex vector space structure from (6.2), then multiplication by complex scalars commutes with the Lie bracket, since

$$i\,[x, y] = J\,[x, y] = [Jx, y] = [ix, y] = [x, Jy] = [x, iy].$$

Hence, $\mathfrak{g}$ has the structure of a complex Lie algebra. Of course, the Lie bracket of any complex Lie algebra is compatible with its complex structure.

A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. (A real form of $\mathfrak{g}$ (as vector space) is not necessarily a real Lie subalgebra of $\mathfrak{g}$. For example, $\mathbb{R}e + \mathbb{R}f + \mathbb{R}ih$ is a real form of the vector space $\mathfrak{sl}(2, \mathbb{C})$ but is not a real Lie subalgebra.) It is easy to check that if τ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then $\tau[x, y] = [\tau x, \tau y]$, for all $x, y \in \mathfrak{g}$.

Exercise 6.2.3. Let $\mathrm{u}(n)$ denote the Lie algebra of *skew-Hermitian matrices*; i.e., $\mathrm{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid {}^t\overline{X} = -X\}$. (See Example 2.1.22.) Prove that $\mathrm{u}(n)$ is a real form of $\mathfrak{gl}(n, \mathbb{C})$. If τ denotes the conjugation of $\mathfrak{gl}(n, \mathbb{C})$ with respect to $\mathrm{u}(n)$, show that $\tau(X) = -{}^t\overline{X}$ for all $X \in \mathfrak{gl}(n, \mathbb{C})$.

Next, suppose that $\mathfrak{g}_0$ is a real Lie algebra. The Lie bracket in $\mathfrak{g}_0$ can be extended to its vector space complexification $\mathfrak{g} = \mathfrak{g}_0^c = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$ via

$$[x_1 + Jx_2, y_1 + Jy_2] = [x_1, y_1] - [x_2, y_2] + J([x_1, y_2] + [x_2, y_1]).$$

The operation above is $\mathbb{R}$-bilinear and can be routinely verified to be anticommutative and to satisfy the Jacobi identity. Moreover

$$\begin{aligned} J[x_1 + Jx_2, y_1 + Jy_2] &= -([x_1, y_2] + [x_2, y_1]) + J([x_1, y_1] - [x_2, y_2]) \\ &= [-x_2 + Jx_1, y_1 + Jy_2] \\ &= [J(x_1 + Jx_2), y_1 + Jy_2], \end{aligned}$$

and so it follows that this extension of the Lie bracket to $\mathfrak{g}$ is $\mathbb{C}$-bilinear. Thus the complexification $\mathfrak{g}$ has the structure of a complex Lie algebra, and of course, $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$.

Example 6.2.4. It is obvious that $\mathfrak{gl}(n,\mathbb{R})$ is a real form of $\mathfrak{gl}(n,\mathbb{C})$. We can express this as $(\mathfrak{gl}(\mathbb{R}^n))^c = \mathfrak{gl}(\mathbb{C}^n)$. Now any real vector space V is (duh) a real form of its complexification V^c. If we fix a basis B of V, then the map $T \mapsto M_{B,B}(T)$ identifies $\mathfrak{gl}(V)$ with $\mathfrak{gl}(n,\mathbb{R})$. Complexifying this identification, we see that $(\mathfrak{gl}(V))^c \cong (\mathfrak{gl}(\mathbb{R}^n))^c \cong \mathfrak{gl}(\mathbb{C}^n) \cong \mathfrak{gl}(V^c)$. This identification of $(\mathfrak{gl}(V))^c$ with $\mathfrak{gl}(V^c)$ is concretely given by

$$(T_1 + iT_2)\,(v_1 + iv_2) = T_1v_1 - T_2v_2 + i(T_1v_2 + T_2v_1),$$

for all T_1, $T_2 \in \mathfrak{gl}(V)$ and all v_1, $v_2 \in V$.

If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, then its complexification $\mathfrak{g}^c$ may be viewed as a complex Lie subalgebra of $\mathfrak{gl}(V^c)$.

Exercise 6.2.5. Let $\mathfrak{g}$ be a complex Lie algebra, and let $\mathfrak{g}_R$ denote $\mathfrak{g}$, considered as a real Lie algebra. If B and B_R are the Killing forms on $\mathfrak{g}$ and $\mathfrak{g}_R$, respectively, show that $B_R(x,y) = 2\,\mathrm{Re}\,(B(x,y)$, for all $x, y \in \mathfrak{g}$. Then show that B is nondegenerate $\iff$ B_R is nondegenerate.

(*Note: If you don't know anything about Lie groups, you may safely skip this paragraph.*) A real Lie algebra $\mathfrak{u}$ is said to be *compact* if $\mathfrak{u}$ is the Lie algebra of a compact Lie goup U. Here are two interesting and useful facts about compact Lie algebras (cf. [**?**], Chapter 3):

1. If $\mathfrak{u}$ is compact, then $\mathfrak{u} = \mathfrak{c} \oplus [\mathfrak{u},\mathfrak{u}]$, where $\mathfrak{c}$ is the center of $\mathfrak{u}$ and $[\mathfrak{u},\mathfrak{u}]$ is compact and semisimple.
2. Any complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form $\mathfrak{u}$. This remarkable fact is a cornerstone of representation theory. In a later section, we will consider how to obtain such a real form.

6.3 Cartan's Criterion for Solvability

Lemma 6.3.1. *Let V be a vector space over $\mathbb{C}$, and let $X \in \mathfrak{gl}(V)$. If X is semisimple, then so is $\mathrm{ad}\,X$. If X is nilpotent, then so is $\mathrm{ad}\,X$.*

Proof. If X is nilpotent, then so is $\mathrm{ad}\,X$ by Step 1 in the proof of Lemma 5.2.2.

Suppose that X is semisimple. Let $B = (v_1, \ldots, v_n)$ be a basis of of V consisting of eigenvectors of X, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. Next, we abuse notation and let, for any i, j in $\{1, \ldots, n\}$, $\mathcal{E}_{ij}$ be the linear operator on V whose matrix with respect to B is the elementary matrix E_{ij}:

$$\mathcal{E}_{ij}(v_k) := \delta_{jk}\, v_i$$

Then $(\mathcal{E}_{ij})_{1\le i,j\le n}$ is a basis of $\mathfrak{gl}(V)$. By matrix multiplication, we see that

$$[X, \mathcal{E}_{ij}] = (\lambda_i - \lambda_j)\,\mathcal{E}_{ij}. \tag{6.3}$$

Thus, each $\mathcal{E}_{ij}$ is an eigenvector of $\operatorname{ad} X$ with eigenvalue $\lambda_i - \lambda_j$, and so $\mathfrak{gl}(V)$ has basis $(\mathcal{E}_{ij})$ consisting of eigenvectors of $\operatorname{ad} X$. Thus $\operatorname{ad} X$ is semisimple. □

Lemma 6.3.2. *Suppose that $X \in \mathfrak{gl}(V)$ has Jordan-Chevalley decomposition $X = X_s + X_n$, with X_s semisimple and X_n nilpotent. Then the Jordan-Chevalley decomposition for $\operatorname{ad} X$ is $\operatorname{ad} X = \operatorname{ad} X_s + \operatorname{ad} X_n$.*

Proof. By the preceding lemma, $\operatorname{ad} X_s$ and $\operatorname{ad} X_n$ are semisimple and nilpotent linear operators on $\mathfrak{gl}(V)$, respectively. Moreover, $\operatorname{ad} X_s$ and $\operatorname{ad} X_n$ commute:

$$\begin{aligned}[\operatorname{ad} X_s, \operatorname{ad} X_n] &= \operatorname{ad}[X_s, X_n] \\ &= \operatorname{ad}(X_s X_n - X_n X_s) \\ &= 0,\end{aligned}$$

since X_s and X_n commute. By the uniqueness of the Jordan-Chevalley decomposition, it follows that $\operatorname{ad} X = \operatorname{ad} X_s + \operatorname{ad} X_n$ is the Jordan-Chevalley decomposition of $\operatorname{ad} X$. □

Lemma 6.3.3. *Let $x_0, x_1, \ldots, x_n$ be distinct numbers in $\mathbb{F}$, and let $C_0, C_1, \ldots, C_n$ be any numbers in $\mathbb{F}$. Then there exists a polynomial $P(x)$ in the variable x, with coefficients in $\mathbb{F}$, of degree $\le n$, such that $P(x_i) = C_i$, for all i.*

Proof. According to the *Lagrange Interpolation Formula*, this polynomial is given by

$$P(x) = \frac{\prod_{i\neq 0}(x - x_i)}{\prod_{i\neq 0}(x_0 - x_i)}\,C_0 \;+\; \frac{\prod_{i\neq 1}(x - x_i)}{\prod_{i\neq 1}(x_1 - x_i)}\,C_1 \;+\; \cdots + \; \frac{\prod_{i\neq n}(x - x_i)}{\prod_{i\neq n}(x_n - x_i)}\,C_n$$

It is easy to see that this $P(x)$ satisfies the properties asserted in the lemma. □

Exercise 6.3.4. (Graduate Exercise.) Show that any polynomial satisfying the conclusion of Lemma 6.3.3 is unique. (*Hint:* The formula above comes from a linear system whose coefficient matrix is Vandermonde.)

The following is a technical lemma whose proof features some "out of the box" thinking.

Lemma 6.3.5. *Let V be a vector space over $\mathbb{C}$, and let $A \subset B$ be subspaces of $\mathfrak{gl}(V)$. Let $\mathfrak{m} = \{X \in \mathfrak{gl}(V) \mid [X, B] \subset A\}$. Suppose that some $X \in \mathfrak{m}$ has the property that $tr(XY) = 0$ for all $Y \in \mathfrak{m}$. Then X is nilpotent.*

Proof. Let $S = (v_1, \ldots, v_n)$ be a Jordan basis of V corresponding to X. If $X = X_s + X_n$ is the Jordan-Chevalley decomposition of X, then S consists of eigenvectors of X_s, and the matrix of X_s with respect to S is diagonal, of the form

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \tag{6.4}$$

The matrix of X_n with respect to S is strictly upper triangular, with some 1's right above the diagonal. We want to show that $\lambda_i = 0$ for all i. This will establish that $X_s = 0$, and so $X = X_n$.

Let $E \subset \mathbb{C}$ be the vector space over $\mathbb{Q}$ (= the rationals) spanned by $\lambda_1, \ldots, \lambda_n$. We will show that $E = \{0\}$. This will, of course, show that each $\lambda_i = 0$. If E^* denotes the dual space (over $\mathbb{Q}$) of E, standard linear algebra says that $\dim_{\mathbb{Q}} E^* = \dim_{\mathbb{Q}} E$. (See Subsection 1.3.) Thus it's sufficient to prove that $E^* = \{0\}$. That is, we will prove that any $\mathbb{Q}$-linear functional on E must vanish identically.

So let $f \in E^*$. Then let $Y \in \mathfrak{gl}(V)$ be the linear map on V whose matrix with respect to the basis S above is the diagonal matrix

$$\begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}$$

We will prove that $\operatorname{ad} Y$ is a polynomial in $\operatorname{ad} X$, with zero constant term, using the interpolation result established above.

By equation (6.3), the semisimple operator X_s satisfies

$$\operatorname{ad} X_s\,(E_{ij}) = (\lambda_i - \lambda_j)\,E_{ij}. \tag{6.5}$$

For the same reason, the semisimple operator Y satisfies

$$\operatorname{ad} Y\,(E_{ij}) = (f(\lambda_i) - f(\lambda_j))\,E_{ij}. \tag{6.6}$$

According to Lemma 6.3.3, there exists a polynomial $G(x)$ in the variable x, with complex coefficients, such that

$$\begin{aligned} G(0) &= 0, \\ G(\lambda_i) &= f(\lambda_i) && \text{for all } i = 1, \ldots, n \\ G(\lambda_i - \lambda_j) &= f(\lambda_i) - f(\lambda_j) && \text{for all } i, j = 1, \ldots, n \end{aligned}$$

This polynomial is well-defined, since if $\lambda_i - \lambda_j = \lambda_k - \lambda_l$, then $f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j) = f(\lambda_k - \lambda_l) = f(\lambda_k) - f(\lambda_l)$. There are at most $2\binom{n}{2} + n + 1 = n^2 + 1$

elements in the set $\{0\} \cup \{\lambda_i\}_{i=1}^n \cup \{\lambda_i - \lambda_j\}_{i,j=1}^n$, so $G(x)$ can be assumed to have degree $\leq n^2$, but this does not matter.

Let us now compute the linear operator $G(\operatorname{ad} X_s)$ on $\mathfrak{gl}(V)$. It suffices to do this on each elementary matrix E_{ij}. Now, by equation (6.5), $\operatorname{ad} X_s\,(E_{ij}) = (\lambda_i - \lambda_j)\, E_{ij}$, and so

$$\begin{aligned} G(\operatorname{ad} X_s)\,(E_{ij}) &= G(\lambda_i - \lambda_j)\, E_{ij} \\ &= (f(\lambda_i) - f(\lambda_j))\, E_{ij} \\ &= \operatorname{ad} Y\,(E_{ij}), \end{aligned}$$

the last equation coming from (6.6). It follows that $G(\operatorname{ad} X_s) = \operatorname{ad} Y$. (Note also that the condition $G(\lambda_i) = f(\lambda_i)$ for all i implies that $G(X_s) = Y$.)

Now by Lemma 6.3.2, the semisimple part of $\operatorname{ad} X$ is $\operatorname{ad} X_s$, which by Theorem 1.9.14 is a polynomial in $\operatorname{ad} X$ with zero constant term. Since the polynomial $G(x)$ also has zero constant term, we see that $\operatorname{ad} Y = G(\operatorname{ad} X_s)$ is a polynomial in $\operatorname{ad} X$ with zero constant term:

$$\operatorname{ad} Y = a_r\,(\operatorname{ad} X)^r + a_{r-1}\,(\operatorname{ad} X)^{r-1} + \cdots + a_1\,\operatorname{ad} X.$$

Since, by hypothesis, $\operatorname{ad} X\,(B) \subset A$, it follows from the above (and the fact that $A \subset B$) that $\operatorname{ad} Y\,(B) \subset A$. Therefore, by the definition of $\mathfrak{m}$, we see that $Y \in \mathfrak{m}$.

Now by the hypothesis on X, we have $\operatorname{tr}(XY) = 0$. With respect to the basis S of V, the product XY has matrix

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} = \begin{pmatrix} \lambda_1 f(\lambda_1) & & * \\ & \ddots & \\ 0 & & \lambda_n f(\lambda_n) \end{pmatrix}$$

and so $\operatorname{tr}(XY) = \sum_{i=1}^n \lambda_i f(\lambda_i)$.

Thus $0 = \sum_{i=1}^n \lambda_i f(\lambda_i)$. Applying the linear functional f to this equality, we get $0 = \sum_{i=1}^n f(\lambda_i)^2$. Since the $f(\lambda_i)$ are all in $\mathbb{Q}$, we conclude that $f(\lambda_i) = 0$ for all i. Thus $f = 0$, so $E^* = \{0\}$, so $E = \{0\}$, and so $\lambda_i = 0$ for all i.

We conclude that $X = X_n$, and the lemma is proved. □

Lemma 6.3.6. *Let V be a vector space over $\mathbb{F}$. If $X, Y, Z \in \mathfrak{gl}(V)$, then $tr([X,Y]Z) = tr(X[Y,Z])$.*

This follows from

$$\begin{aligned}\operatorname{tr}([X,Y]Z) &= \operatorname{tr}((XY-YX)Z)\\ &= \operatorname{tr}(XYZ-YXZ)\\ &= \operatorname{tr}(XYZ)-\operatorname{tr}(YXZ)\\ &= \operatorname{tr}(XYZ)-\operatorname{tr}(XZY)\\ &= \operatorname{tr}(X(YZ-ZY))\\ &= \operatorname{tr}(X[Y,Z]).\end{aligned}$$

The following theorem gives the version of Cartan's solvability criterion (Theorem 6.1.3) for Lie subalgebras of $\mathfrak{gl}(V)$.

Theorem 6.3.7. *(Cartan's Criterion for $\mathfrak{gl}(V)$, V complex.) Let V be a vector space over $\mathbb{C}$, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if $tr(XY)=0$ for all $X \in [\mathfrak{g},\mathfrak{g}]$ and all $Y \in \mathfrak{g}$.*

Proof. Suppose that $\mathfrak{g}$ is solvable. Then by Lie's Theorem (Theorem 4.2.3), there is a basis S of V relative to which every element of $\mathfrak{g}$ has an upper triangular matrix. It follows that every element of $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]$ has a strictly upper triangular matrix relative to this basis. If $X \in \mathfrak{g}'$ and $Y \in \mathfrak{g}$, it is easy to see that the matrix of XY with respect to S is also strictly upper triangular. Thus $\operatorname{tr}(XY)=0$.

Conversely, suppose that $\underline{\operatorname{tr}(XY)=0 \text{ for all } X\in\mathfrak{g}' \text{ and } Y\in\mathfrak{g}}$. We want to prove that $\mathfrak{g}$ is solvable. By Theorem 5.2.5, it suffices to prove that $\mathfrak{g}'$ is nilpotent. For this, it suffices in turn to prove that every $X \in \mathfrak{g}'$ is a nilpotent linear operator on V. For then, by Step 1 in the proof of Engel's Lemma (Lemma 5.2.2), $\operatorname{ad} X$ is nilpotent, and so by Engel's Theorem (Theorem 5.2.1), $\mathfrak{g}'$ is nilpotent.

To this end, we will use Lemma 6.3.5 with $A=\mathfrak{g}'$ and $B=\mathfrak{g}$. The subspace $\mathfrak{m}$ will then be $\{Y \in \mathfrak{gl}(V) \mid [Y,\mathfrak{g}] \subset \mathfrak{g}'\}$. Clearly, $\mathfrak{g}' \subset \mathfrak{m}$. (In fact, $\mathfrak{g} \subset \mathfrak{m}$.) To apply the lemma, we will need to prove that $\operatorname{tr}(XY)=0$ for all $X \in \mathfrak{g}'$ and all $Y \in \mathfrak{m}$.

Now $\mathfrak{g}'$ is spanned by the brackets $[Z,W]$, for all $Z, W \in \mathfrak{g}$. Suppose that $Y \in \mathfrak{m}$. By Lemma 6.3.6, $\operatorname{tr}([Z,W]Y)=\operatorname{tr}(Z[W,Y])=\operatorname{tr}([W,Y]Z)$. But $[W,Y] \in \mathfrak{g}'$ (by the definition of $\mathfrak{m}$). So, by our underlined hypothesis above, $\operatorname{tr}([W,Y]Z)=0$.

This shows that $\operatorname{tr}(XY)=0$ for all generators $X=[Z,W]$ of $\mathfrak{g}'$ and all $Y \in \mathfrak{m}$. Since the trace is linear, we conclude that $\operatorname{tr}(XY)=0$ for all $X \in \mathfrak{g}'$ and $Y \in \mathfrak{m}$. Hence, by Lemma 6.3.5, each $X \in \mathfrak{g}'$ is nilpotent, and the theorem is proved. $\square$

Corollary 6.3.8. *(Cartan's Criterion for $\mathfrak{gl}(V)$, V real.) Let V be a vector space over $\mathbb{R}$, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if $tr\,(XY) = 0$ for all $X \in \mathfrak{g}'$ and all $Y \in \mathfrak{g}$.*

Proof. Let $B = (v_1, \dots, v_n)$ be a fixed basis of V. Then B is also a complex basis of V^c. As remarked earlier, the map $T \mapsto M_{B,B}(T)$ is a Lie algebra isomorphism of $\mathfrak{gl}(V)$ onto $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(V^c)$ onto $\mathfrak{gl}(n, \mathbb{C})$. Thus it suffices to prove this corollary for Lie subalgebras $\mathfrak{g}$ of $\mathfrak{gl}(n, \mathbb{R})$.

The derived algebra $(\mathfrak{g}^c)'$ is the linear span of elements of the form $[X_1+iX_2, Y_1+iY_2] = [X_1, X_2]-[Y_1, Y_2]+i([X_1, Y_2]+[X_2, Y_1])$, where the X_i, $Y_j \in \mathfrak{g}$, and from this it is not hard to see that $(\mathfrak{g}^c)' = (\mathfrak{g}')^c$. By induction, we conclude that $(\mathfrak{g}^c)^{(r)} = (\mathfrak{g}^{(r)})^c$. This in turn shows that $\mathfrak{g}$ is solvable $\iff$ $\mathfrak{g}^c$ is solvable.

Now if $\mathfrak{g}$ is solvable, then $\mathfrak{g}^c$ is a solvable Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. Hence by Theorem 6.3.7, $\operatorname{tr}(XY) = 0$ for all $X \in (\mathfrak{g}^c)'$ and all $Y \in \mathfrak{g}^c$. In particular, $\operatorname{tr}(XY) = 0$ for all $X \in \mathfrak{g}'$ and $Y \in \mathfrak{g}$.

Conversely, suppose that $\operatorname{tr}(XY) = 0$ for all $X \in \mathfrak{g}'$ and $Y \in \mathfrak{g}$. We want to prove that $\operatorname{tr}(ZW) = 0$ for all $Z \in (\mathfrak{g}^c)'$ and all $W \in \mathfrak{g}^c$. But then we can resolve Z into its real and imaginary components: $Z = X_1 + iX_2$, where X_1, $X_2 \in \mathfrak{g}'$. Likewise, $W = Y_1 + iY_2$, with Y_1, $Y_2 \in \mathfrak{g}$. Thus $\operatorname{tr}(ZW) = \operatorname{tr}(X_1Y_1 - X_2Y_2) + i\operatorname{tr}(X_1Y_2 + X_2Y_1) = 0$. By Theorem 6.3.7, we conclude that $\mathfrak{g}^c$ is solvable, and hence $\mathfrak{g}$ also is. □

We are now ready to prove Cartan's criterion for solvability.

Proof of Theorem 6.1.3: Suppose first that $\mathfrak{g}$ is solvable. Then $\operatorname{ad}\mathfrak{g}$ is solvable as the homomorphic image of a solvable algebra. Thus $\operatorname{ad}\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{gl}(\mathfrak{g})$, and so by the "only if" part of Theorem 6.3.7 (for $\mathbb{F} = \mathbb{C}$) or Corollary 6.3.8 (for $\mathbb{F} = \mathbb{R}$), we conclude that $B(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and all $y \in \mathfrak{g}$.

Conversely, suppose that $B(x, y) = 0$ for all $x \in \mathfrak{g}'$ and all $y \in \mathfrak{g}$. This translates to the condition that $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $\operatorname{ad} x \in \operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ and all $\operatorname{ad} y \in \operatorname{ad}\mathfrak{g}$. By the "if" part of Theorem 6.3.7 or Corollary 6.3.8, we conclude that $\operatorname{ad}\mathfrak{g}$ is a solvable subalgebra of $\mathfrak{gl}(\mathfrak{g})$. But $\operatorname{ad}\mathfrak{g} = \mathfrak{g}/\mathfrak{c}$, so, since $\mathfrak{c}$ is obviously solvable, Proposition 4.1.10 implies that $\mathfrak{g}$ is solvable. □

The argument above can be summarized in the following string of equivalences, using Proposition 4.1.10 in the first step:

$\mathfrak{g}$ is solvable $\iff$ $\operatorname{ad}\mathfrak{g} \cong \mathfrak{g}/\mathfrak{c}$ is a solvable subalgebra of $\mathfrak{gl}(\mathfrak{g})$ $\iff$ $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $\operatorname{ad} x \in (\operatorname{ad}\mathfrak{g})'$ and all $\operatorname{ad} y \in \operatorname{ad}\mathfrak{g}$ $\iff$ $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $x \in \mathfrak{g}'$ and all $y \in \mathfrak{g}$ $\iff$ $B(x, y) = 0$ for all $x \in \mathfrak{g}'$ and all $y \in \mathfrak{g}$.

6.4 Cartan's Criterion for Semisimplicity

We've already seen that the Killing form B on a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is an $\mathbb{F}$-valued symmetric bilinear form. The following lemma gives an invariance property satisfied by B:

Lemma 6.4.1. *The Killing form B satisfies the property that*

$$B([x,y],z) = B(x,[y,z]) \tag{6.7}$$

for all x, y, $z \in \mathfrak{g}$.

Proof. By Lemma 6.3.6, we have

$$\mathrm{tr}\,([\mathrm{ad}\,x, \mathrm{ad}\,y] \circ \mathrm{ad}\,z) = \mathrm{tr}\,(\mathrm{ad}\,x \circ [\mathrm{ad}\,y, \mathrm{ad}\,z]),$$

and so

$$\mathrm{tr}\,(\mathrm{ad}[x,y] \circ \mathrm{ad}\,z) = \mathrm{tr}\,(\mathrm{ad}\,x \circ \mathrm{ad}[y,z]),$$

which implies the result. □

The *radical* of B is the subspace of $\mathfrak{g}$ given by $\mathfrak{g}^\perp := \{x \in \mathfrak{g} \,|\, B(x,y) = 0 \text{ for all } y \in \mathfrak{g}\}$. Note that by Theorem 1.10.5, B is nondegenerate if and only if $\mathfrak{g}^\perp = \{0\}$.

Corollary 6.4.2. *$\mathfrak{g}^\perp$ is an ideal of $\mathfrak{g}$.*

Proof. Let $x \in \mathfrak{g}^\perp$. Then for all $y \in \mathfrak{g}$, we claim that $[x,y] \in \mathfrak{g}^\perp$. This is easy: for any $z \in \mathfrak{g}$, we have $B([x,y],z) = B(x,[y,z]) = 0$, so it follows that $[x,y] \in \mathfrak{g}^\perp$ □

Proposition 6.4.3. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. If $B_\mathfrak{a}$ denotes the Killing form of the Lie algebra $\mathfrak{a}$, then $B_\mathfrak{a}(x,y) = B(x,y)$ for all x, $y \in \mathfrak{a}$. Thus, $B_\mathfrak{a}$ equals the restriction of B to $\mathfrak{a} \times \mathfrak{a}$.*

Proof. Let $\mathfrak{r}$ be any subspace of $\mathfrak{g}$ complementary to $\mathfrak{a}$, so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{r}$. Next let B' and B'' be bases of $\mathfrak{a}$ and $\mathfrak{r}$, respectively. If x and y belong to $\mathfrak{a}$, then $\mathrm{ad}\,x$ and $\mathrm{ad}\,y$ both map $\mathfrak{g}$ to $\mathfrak{a}$; thus, relative to the basis (B', B'') of $\mathfrak{g}$, the matrices of $\mathrm{ad}\,x$ and $\mathrm{ad}\,y$ have block form

$$\mathrm{ad}\,x = \begin{pmatrix} R_1 & S_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}\,y = \begin{pmatrix} R_2 & S_2 \\ 0 & 0 \end{pmatrix},$$

respectively. In the above, R_1 is the matrix of the restriction $\operatorname{ad} x|_{\mathfrak{a}}$ with respect to the basis B' of $\mathfrak{a}$. Likewise, R_2 is the matrix of $\operatorname{ad} y|_{\mathfrak{a}}$ with respect to B'. Hence

$$\begin{aligned} B(x,y) &= \operatorname{tr}\, \operatorname{ad} x \circ \operatorname{ad} y \\ &= \operatorname{tr} \begin{pmatrix} R_1R_2 & R_1S_2 \\ 0 & 0 \end{pmatrix} \\ &= \operatorname{tr}(R_1R_2) \\ &= \operatorname{tr}\, \operatorname{ad} x|_{\mathfrak{a}} \circ \operatorname{ad} y|_{\mathfrak{a}} \\ &= B_{\mathfrak{a}}(x,y). \end{aligned}$$

□

Example 6.4.4. In Example 3.4.1, we saw that for the basis (e, f, h) of $\mathfrak{sl}(2,\mathbb{C})$, we could represent $\operatorname{ad} e$, $\operatorname{ad} f$, and $\operatorname{ad} h$ by the following matrices relative to this basis:

$$\operatorname{ad} e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \operatorname{ad} f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \ \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$\begin{aligned} B(e,e) &= \operatorname{tr}(\operatorname{ad} e \circ \operatorname{ad} e) = \operatorname{tr} \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \\ B(e,f) &= \operatorname{tr}(\operatorname{ad} e \circ \operatorname{ad} f) = \operatorname{tr} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 4 \\ B(e,h) &= \operatorname{tr}(\operatorname{ad} e \circ \operatorname{ad} h) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = 0 \\ B(f,f) &= \operatorname{tr}(\operatorname{ad} f \circ \operatorname{ad} f) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \\ B(f,h) &= \operatorname{tr}(\operatorname{ad} f \circ \operatorname{ad} f) = \operatorname{tr} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 0 \\ B(h,h) &= \operatorname{tr}(\operatorname{ad} f \circ \operatorname{ad} f) = \operatorname{tr} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8 \end{aligned}$$

Thus the matrix of the bilinear form B with respect to $(e,\ f,\ h)$ is

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

The determinant of this matrix is $-128 \neq 0$. Thus, by Theorem 1.10.5, B is nondegenerate.

Lemma 6.4.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$, and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then the derived series $\mathfrak{a} \supset \mathfrak{a}' \supset \mathfrak{a}^{(2)} \supset \cdots$ consists of ideals of $\mathfrak{g}$.*

Proof. This is an easy induction. Certainly, $\mathfrak{a} = \mathfrak{a}^{(0)}$ is an ideal of $\mathfrak{g}$ by hypothesis. Then, assuming that $\mathfrak{a}^{(r)}$ is an ideal of $\mathfrak{g}$, we have

$$\begin{aligned}
[\mathfrak{a}^{(r+1)}, \mathfrak{g}] &= [[\mathfrak{a}^{(r)}, \mathfrak{a}^{(r)}], \mathfrak{g}] \\
&= [[\mathfrak{a}^{(r)}, \mathfrak{g}], \mathfrak{a}^{(r)}] && \text{(by the Jacobi identity)} \\
&\subset [\mathfrak{a}^{(r)}, \mathfrak{a}^{(r)}] && \text{(by the induction hypothesis)} \\
&= \mathfrak{a}^{(r+1)},
\end{aligned}$$

so $\mathfrak{a}^{(r+1)}$ is an ideal of $\mathfrak{g}$. □

Lemma 6.4.6. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ has no abelian ideals $\mathfrak{a} \neq \{0\}$.*

Proof. Suppose that $\mathfrak{g}$ is semisimple. Any abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ is solvable, so $\mathfrak{a} \subset \mathcal{R}_s = \{0\}$, and thus $\mathfrak{a} = \{0\}$.

Conversely, suppose that $\mathfrak{g}$ is not semisimple. Then the solvable radical $\mathcal{R}_s \neq \{0\}$. Let $\mathcal{R}_s \supsetneq \mathcal{R}_s' \supsetneq \cdots \supsetneq \mathcal{R}_s^{(k)} \supsetneq \{0\}$ be the derived series for $\mathcal{R}_s$. By the preceding lemma, each of the $\mathcal{R}_s^{(i)}$ is an ideal of $\mathfrak{g}$. The last non-zero ideal $\mathcal{R}_s^{(k)}$ is thus a non-zero abelian ideal of $\mathfrak{g}$. Thus, $\mathfrak{g}$ has non-zero abelian ideals. □

We are now ready to prove Cartan's criterion for semisimplicity:

Proof of Theorem 6.1.4: To avoid the obvious triviality, we may assume that $\mathfrak{g} \neq \{0\}$. Suppose first that $\mathfrak{g}$ is a Lie algebra over $\mathbb{F}$ such that B is nondegenerate. To prove that $\mathfrak{g}$ is semisimple, it suffices, by Lemma 6.4.6, to prove that $\mathfrak{g}$ has no non-zero abelian ideals. Suppose that $\mathfrak{a}$ is an abelian ideal. then for $x \in \mathfrak{a}$ and y, $z \in \mathfrak{g}$, we have

$$(\operatorname{ad} x \circ \operatorname{ad} y)^2 (z) = [x, [y, [x, [y, z]]]] \in [\mathfrak{a}, \mathfrak{a}] = \{0\},$$

so $(\operatorname{ad} x \circ \operatorname{ad} y)^2 = 0$. Thus, $\operatorname{ad} x \circ \operatorname{ad} y$ is nilpotent. This implies that $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$. (See equation 1.17.) Hence $B(x, y) = 0$ for all $x \in \mathfrak{a}$ and all $y \in \mathfrak{g}$. Therefore, $\mathfrak{a} \subset \mathfrak{g}^\perp = \{0\}$, and so $\mathfrak{a} = \{0\}$. Hence any abelian ideal of $\mathfrak{g}$ is $\{0\}$, and so $\mathfrak{g}$ is semisimple.

Conversely, suppose that $\mathfrak{g}$ is semisimple. We need to show in this case that $\mathfrak{g}^\perp = \{0\}$. Now by definition, $B(x, y) = 0$ for all $x \in \mathfrak{g}^\perp$ and $y \in \mathfrak{g}$. Hence

$B(x, y) = 0$ for all $x \in \mathfrak{g}^\perp$ and $y \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$. By Proposition 6.4.3, we see that $B_{\mathfrak{g}^\perp}(x, y) = 0$ for all $x \in \mathfrak{g}^\perp$ and $y \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$. Then by Cartan's solvability criterion (Theorem 6.1.3), we see that $\mathfrak{g}^\perp$ is solvable. Since it is an ideal by Corollary 6.4.2, it follows that $\mathfrak{a} \subset \mathcal{R}_s = \{0\}$, and so B is nondegenerate. □

Corollary 6.4.7. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Then $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g}^c$ is semisimple.*

Proof. Let B and B^c denote the Killing forms on $\mathfrak{g}$ and $\mathfrak{g}^c$, respectively. Then it suffices to prove that B is nondegenerate $\iff$ B^c is nondegenerate. Note that $B^c(X, Y) = B(X, Y)$ if X and Y are in $\mathfrak{g}$.

Using this last observation, it is not hard to see that $(\mathfrak{g}^\perp)^c = (\mathfrak{g}^c)^\perp$, so $\mathfrak{g}^\perp = \{0\} \iff (\mathfrak{g}^c)^\perp = \{0\}$. □

Exercise 6.4.8. Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$. Let $\mathfrak{g}_\mathbb{R}$ be the Lie algebra $\mathfrak{g}$ considered as a real Lie algebra. Prove that $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g}_\mathbb{R}$ is semisimple.

We next consider a slight variant of the Killing form, called the *trace form*, on Lie algebras of linear operators.

Let V be a vector space over $\mathbb{F}$, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. The *trace form* on $\mathfrak{g}$ is the symmetric bilinear form $(X, Y) \mapsto \operatorname{tr}(XY)$, for all X, $Y \in \mathfrak{g}$.

Proposition 6.4.9. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\mathfrak{g}$ is semisimple, then its trace form is nondegenerate.*

Proof. The proof is similar to that of Theorem 6.1.4. Let $\mathcal{I} = \{X \in \mathfrak{g} \mid \operatorname{tr}(XY) = 0 \text{ for all } Y \in \mathfrak{g}\}$. Then it follows easily from Lemma 6.3.6 that $\mathcal{I}$ is an ideal of $\mathfrak{g}$.

For any $X \in [\mathcal{I}, \mathcal{I}]$ and $Y \in \mathcal{I}$, we have $\operatorname{tr}(XY) = 0$. Hence by Theorem 6.3.7 and Corollary 6.3.8, $\mathcal{I}$ is solvable, and so $\mathcal{I} = \{0\}$. Thus the trace form is nondegenerate. □

Exercise 6.4.10. Is the converse true? Explicitly, suppose that the trace form on $\mathfrak{g} \subset \mathfrak{gl}(V)$ is nondegenerate. Is $\mathfrak{g}$ semisimple? Prove or give a counterexample.

We conclude this section by calculating the Killing form on the Lie algebras $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{R})$, and $\mathfrak{sl}(n, \mathbb{R})$.

Example 6.4.11. (The Killing form on $\mathfrak{gl}(n, \mathbb{C})$.) This example requires a little background (just a tiny bit!) in basic analysis. We will use the steps below to prove that the Killing form on $\mathfrak{gl}(n, \mathbb{C})$ is given by the formula

$$B(X, Y) = 2n \operatorname{tr}(XY) - 2 \operatorname{tr} X \cdot \operatorname{tr} Y \qquad (X,\, Y \in \mathfrak{gl}(n, \mathbb{C})) \tag{6.8}$$

1. Let $\mathcal{O} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid$ each eigenvalue of X has multiplicity $1\}$. We claim that $\mathcal{O}$ is dense in $\mathfrak{gl}(n, \mathbb{C})$.

   To prove this, recall that $\mathfrak{gl}(n, \mathbb{C})$ is just $\mathbb{C}^{n^2} = \mathbb{C}^{n \times n}$. The standard inner product is just the Hilbert-Schmidt inner product:

   $$\langle X, Y \rangle = \operatorname{tr}(X\, {}^t\overline{Y}) \qquad (X, Y \in \mathfrak{gl}(n, \mathbb{C}).)$$

   (See Example 1.11.2.) The resulting (standard) metric is

   $$\|X - Y\| = \left\{ \sum_{j=1}^{n} \sum_{k=1}^{n} |X_{jk} - Y_{jk}|^2 \right\}^{1/2}.$$

   If T is a fixed non-singular $n \times n$ complex matrix, then the conjugation map $c_T : X \mapsto TXT^{-1}$ is a linear operator on $\mathfrak{gl}(n, \mathbb{C})$, and hence is continuous. Since c_T is bijective (with inverse $c_{T^{-1}}$), it is a homeomorphism. Moreover, since the eigenvalues of a matrix are preserved under conjugation, we see that $c_T(\mathcal{O}) = \mathcal{O}$.

   Suppose now that $X \in \mathfrak{gl}(n, \mathbb{C})$. Then by Theorem 1.6.2, X is conjugate to an upper triangular matrix Y. There is a sequence $\{Y_l\}$ of upper triangular matrices converging to Y such that each Y_l has distinct diagonal entries. (The matrices Y_l are obtained by slightly perturbing the diagonal entries of Y.) This is a sequence in $\mathcal{O}$ converging to Y, and by taking the inverse conjugation, we obtain a sequence $\{X_l\}$ in $\mathcal{O}$ converging to X. This proves that $\mathcal{O}$ is dense in $\mathfrak{gl}(n, \mathbb{C})$.

2. By the Jordan canonical form (Theorem 1.8.5) (or even just the general block diagonal form in Proposition 1.8.2), each matrix in $\mathcal{O}$ is diagonalizable. Thus the diagonalizable matrices are dense in $\mathfrak{gl}(n, \mathbb{C})$.

   (This is not true, by the way, in $\mathfrak{gl}(n, \mathbb{R})$. That is, the set of matrices in $\mathfrak{gl}(n, \mathbb{R})$ conjugate to a real diagonal matrix is not dense. To see this, note that the coefficients of the characteristic polynomial of a matrix $X \in \mathfrak{gl}(n, \mathbb{R})$ are polynomials in the entries of X. Thus a slight perturbation of the matrix entries in X will result in a slight perturbation of its characteristic polynomial. If the characteristic polynomial of X has an irreducible quadratic factor, so does the characteristic polynomial of any slight perturbation of X.)

3. We will show that

   $$B(H, H) = 2n \operatorname{tr}(H^2) - 2(\operatorname{tr} H)^2 \tag{6.9}$$

   for any diagonalizable $H \in \mathfrak{gl}(n, \mathbb{C})$. Since the diagonalizable matrices are dense and since the trace function is continuous, (6.9) will then hold for any $H \in \mathfrak{gl}(n, \mathbb{C})$. Polarizing (6.9) (i.e., replacing H by $X + Y$ and $X - Y$ and then adding the resulting equalities), we can conclude that the trace formula (6.8) holds for all $X, Y \in \mathfrak{gl}(n, \mathbb{C})$.

Let us now prove (6.9). Since $B(H,H)$, $\operatorname{tr} H$, and $\operatorname{tr}(H^2)$ are all unchanged when H is replaced by any of its conjugates, we can assume that H is diagonal:

$$H = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}.$$

Let E_{jk} be the elementary $n \times n$ matrix whose (j,k)-entry is 1, and all of whose other entries are 0. Then a simple computation shows that

$$\operatorname{ad} H(E_{jk}) = (h_j - h_k)E_{jk}$$

so the set $\{E_{jk}\}$ is a basis of $\mathfrak{gl}(n,\mathbb{C})$ consisting of eigenvectors of $\operatorname{ad} H$. Hence

$$\begin{aligned} B(H,H) &= \operatorname{tr}(\operatorname{ad} H)^2 \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}(h_j - h_k)^2 \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}(h_j^2 + h_k^2) - 2\sum_{j=1}^{n}\sum_{k=1}^{n} h_j h_k \\ &= 2n\sum_{j=1}^{n} h_j^2 - 2\left(\sum_{j=1}^{n} h_j\right)\left(\sum_{k=1}^{n} h_k\right) \\ &= 2n \operatorname{tr}(H^2) - 2\left(\operatorname{tr} H\right)^2. \end{aligned}$$

This proves (6.9), and the Killing form trace formula (6.8).

In the example above, we can avoid appealing to analysis by observing that the real Lie algebra $\mathrm{u}(n)$ of (complex) $n \times n$ skew-Hermitian matrices is a real form of $\mathrm{gl}(n,\mathbb{C})$. If $X \in \mathrm{u}(n)$, then X is normal, so there is a unitary $n \times n$ matrix u such that $H = uXu^{-1}$ is diagonal. Note that H also belongs to $\mathrm{u}(n)$. Since H is diagonal, the formula (6.9) for $B(H,H)$ applies. Conjugating back to X, we see that formula (6.9) holds for $B(X,X)$. We then polarize to get formula (6.8) for all X and Y in $\mathrm{u}(n)$, and then use the $\mathbb{C}$-bilinearity of both sides in that formula (as well as the fact that $\mathrm{u}(n)$ is a real form) to conclude that (6.8) in fact holds for all X and Y in $\mathfrak{gl}(n,\mathbb{C})$.

Example 6.4.12. (The Killing form on $\mathfrak{gl}(n,\mathbb{R})$.) The complexification of $\mathfrak{gl}(n,\mathbb{R})$ is (obviously) $\mathfrak{gl}(n,\mathbb{C})$. If B and B^c are the Killing forms on $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{C})$, respectively, the proof of Corollary 6.4.7 shows that $B(X,Y) = B^c(X,Y)$ for all $X, Y \in \mathfrak{gl}(n,\mathbb{R})$. Thus the Killing form on $\mathfrak{gl}(n,\mathbb{R})$ is also given by (6.8).

Example 6.4.13. (The Killing form on $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{R})$.) Since $\operatorname{tr}[X,Y] = 0$ for all X, $Y \in \mathfrak{gl}(n,\mathbb{C})$, we see that $\mathfrak{sl}(n,\mathbb{C})$ is an ideal of $\mathfrak{gl}(n,\mathbb{C})$. Thus the

Killing form on $\mathfrak{sl}(n, \mathbb{C})$ is the restriction of the Killing form on $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{sl}(n, \mathbb{C}) \times \mathfrak{sl}(n, \mathbb{C})$. Formula (6.8)then gives the Killing form on $\mathfrak{sl}(n, \mathbb{C})$:

$$B(X, Y) = 2n \operatorname{tr}(XY) \qquad (X, Y \in \mathfrak{sl}(n, \mathbb{C}).) \tag{6.10}$$

This is also the Killing form on $\mathfrak{sl}(n, \mathbb{R})$.

Later in Section **??**, we will obtain the Killing form on $\mathfrak{sl}(n, \mathbb{C})$ by a method that doesn't use analysis.

Proposition 6.4.14. *The Lie algebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{C})$ are semisimple.*

For $\mathfrak{sl}(n, \mathbb{C})$, it is enough to note that if $X \in \mathfrak{sl}(n, \mathbb{C})$, then so is ${}^t\overline{X}$. Then $B(X, {}^t\overline{X}) = 2n \operatorname{tr}(X\, {}^t\overline{X})$ is the Hilbert-Schmidt norm of X. This easily implies that B is nondegenerate.

By Corollary 6.4.7, $\mathfrak{sl}(n, \mathbb{R})$ is also semisimple. Note that $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{R})$ are not semisimple, since their centers contain the multiples of the identity matrix, hence are non-zero.

According to Example 3.3.7, the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is simple. It turns out (and we will show later!) that the Lie algebras $\mathfrak{sl}(n, \mathbb{F})$ are simple for $n \geq 2$.

Exercise 6.4.15. Show that the center of $\mathfrak{gl}(n, \mathbb{F})$ is one-dimensional.