Chapter 5

Nilpotent Lie Algebras and Engel's Theorem

5.1 Nilpotent Lie Algebras

For any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, we define a sequence of subspaces of $\mathfrak{g}$ as follows. Let $\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^2(\mathfrak{g}) = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, and inductively, let $\mathcal{C}^{i+1}(\mathfrak{g}) = [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}]$ for all i.

Proposition 5.1.1. *The subspaces $\mathcal{C}^i(\mathfrak{g})$ satisfy the following properties:*

1. *Each $\mathcal{C}^i(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$.*
2. $\mathcal{C}^1(\mathfrak{g}) \supset \mathcal{C}^2(\mathfrak{g}) \supset \cdots \supset \mathcal{C}^i(\mathfrak{g}) \supset \cdots$
3. $\mathcal{C}^i(\mathfrak{g})/\mathcal{C}^{i+1}(\mathfrak{g})$ *lies inside the center of* $\mathfrak{g}/\mathcal{C}^{i+1}(\mathfrak{g})$

Proof. We prove (1) by induction on i, the case $i = 1$ being trivial. Suppose that $\mathcal{C}^i(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$. Then

$$[\mathfrak{g}, \mathcal{C}^{i+1}(\mathfrak{g})] = [\mathfrak{g}, [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}]] \subset [\mathfrak{g}, \mathcal{C}^i(\mathfrak{g})] = \mathcal{C}^{i+1}(\mathfrak{g}),$$

proving that $\mathcal{C}^{i+1}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. Moreover, for any derivation D of $\mathfrak{g}$, we have

$$\begin{aligned}
D(\mathcal{C}^{i+1}(\mathfrak{g})) &= D([\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}]) \\
&\subset [D(\mathcal{C}^i(\mathfrak{g})), \mathfrak{g}] + [\mathcal{C}^i(\mathfrak{g}), D\mathfrak{g}] \\
&\subset [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}] + [C^i(\mathfrak{g}), \mathfrak{g}] \\
&= \mathcal{C}^{i+1}(\mathfrak{g}),
\end{aligned}$$

proving that $\mathcal{C}^{i+1}(\mathfrak{g})$ is characteristic.

Note that since $\mathcal{C}^{i+1}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, it is also an ideal of $\mathcal{C}^i(\mathfrak{g})$.

Likewise, for (2), we prove the inclusion $\mathcal{C}^{i+1}(\mathfrak{g}) \subset C^i(\mathfrak{g})$ by induction on i, with the case $i = 1$ corresponding to the trivial inclusion $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Assume, then, that $\mathcal{C}^{i+1}(\mathfrak{g}) \subset C^i(\mathfrak{g})$. Then $\mathcal{C}^{i+2}(\mathfrak{g}) = [\mathcal{C}^{i+1}(\mathfrak{g}), \mathfrak{g}] \subset [\mathcal{C}^i(\mathfrak{g}), \mathfrak{g}] = \mathcal{C}^{i+1}(\mathfrak{g})$.

Finally, for (3), let $x \in \mathcal{C}^i(\mathfrak{g})$. Then for any $y \in \mathfrak{g}$, we have $[x, y] \in \mathcal{C}^{i+1}(\mathfrak{g})$. Hence in the quotient algebra $\mathfrak{g}/\mathcal{C}^{i+1}(\mathfrak{g})$, we have

$$[x + \mathcal{C}^{i+1}(\mathfrak{g}), y + \mathcal{C}^{i+1}(\mathfrak{g})] = [x, y] + \mathcal{C}^{i+1}(\mathfrak{g}) = \mathcal{C}^{i+1}(\mathfrak{g})$$

It follows that every element of $\mathcal{C}^i(\mathfrak{g})/\mathcal{C}^{i+1}(\mathfrak{g})$ is an element of the center of $\mathfrak{g}/\mathcal{C}^{i+1}(\mathfrak{g})$. □

Definition 5.1.2. The *descending central series* for $\mathfrak{g}$ is the sequence of ideals $\mathfrak{g} = \mathcal{C}^1(\mathfrak{g}) \supset \mathcal{C}^2(\mathfrak{g}) \supset \cdots \supset \mathcal{C}^i(\mathfrak{g}) \supset \cdots$. (Since $\dim \mathfrak{g}$ is finite, it is clear that this series stabilizes after some point.) The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k.

Note that the term "central" is appropriate since $\mathcal{C}^i(\mathfrak{g})/\mathcal{C}^{i+1}(\mathfrak{g}) \subset \mathfrak{c}(\mathfrak{g}/\mathcal{C}^{i+1}(\mathfrak{g}))$.

Definition 5.1.3. Let $\mathcal{C}_0 = \{0\}$, $\mathcal{C}_1 = \mathfrak{c}(\mathfrak{g})$ and, recursively, let $\mathcal{C}_i$ be the ideal in $\mathfrak{g}$ such that $\mathcal{C}_i/\mathcal{C}_{i-1} = \mathfrak{c}(\mathfrak{g}/\mathcal{C}_{i-1})$. (This ideal exists and is unique because of the Correspondence Theorem (Theorem 3.2.6).) The *ascending central series* is the sequence of ideals $\{0\} = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_i \subset \cdots$. (Since $\dim \mathfrak{g}$ is finite, this series stabilizes after some point.)

Proposition 5.1.4. *The Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathcal{C}_s = \mathfrak{g}$ for some positive integer s.*

Proof. We may assume that $\mathfrak{g} \neq \{0\}$; otherwise, there is nothing to prove.

Suppose first that $\mathfrak{g}$ is nilpotent. Let k be the smallest (necessarily positive) integer such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$. For any integer i, with $0 \leq i \leq k$, we claim that $\mathcal{C}_i \supset C^{k-i}(\mathfrak{g})$. From this, it will follow that $\mathcal{C}_k \supset \mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$.

To prove the claim, we first note that $\mathcal{C}_0 = \{0\} = \mathcal{C}^k(\mathfrak{g})$, so the claim is certainly true for $k = 0$. Assume, inductively, that $\mathcal{C}_i \supset \mathcal{C}^{k-i}(\mathfrak{g})$. By statement (3) of the Isomorphism Theorem (Theorem 3.2.7) and its proof, the map $\varphi : x + \mathcal{C}^{k-i}(\mathfrak{g}) \mapsto x + \mathcal{C}_i$ is a surjective Lie algebra homomorphism of $\mathfrak{g}/\mathcal{C}^{k-i}(\mathfrak{g})$ onto $\mathfrak{g}/\mathcal{C}_i$, with kernel $C_i/\mathcal{C}^{k-i}(\mathfrak{g})$. By Proposition 3.3.3, φ maps the center of $\mathfrak{g}/\mathcal{C}^{k-i}(\mathfrak{g})$ into the center $\mathcal{C}_{i+1}/\mathcal{C}_i$ of $\mathfrak{g}/\mathcal{C}_i$. But by part (3) of Proposition 5.1.1, $\mathcal{C}^{k-i-1}(\mathfrak{g})/\mathcal{C}^{k-i}(\mathfrak{g})$ lies in the center of $\mathfrak{g}/\mathcal{C}^{k-i}(\mathfrak{g})$. Hence φ maps $\mathcal{C}^{k-i-1}(\mathfrak{g})/\mathcal{C}^{k-i}(\mathfrak{g})$ into $\mathcal{C}_{i+1}/\mathcal{C}_i$.

Thus, if $x \in \mathcal{C}^{k-i-1}(\mathfrak{g})$, then $x + \mathcal{C}_i \in \mathcal{C}_{i+1}/\mathcal{C}_i$, and hence $x \in \mathcal{C}_{i+1}$. This shows that $\mathcal{C}^{k-i-1}(\mathfrak{g}) \subset \mathcal{C}_{i+1}$, completing the induction and proving the claim.

Next we assume that $\mathcal{C}_s = \mathfrak{g}$, for some s. Let k be the smallest integer such that $\mathcal{C}_k = \mathfrak{g}$. Since $\mathfrak{g} \neq \{0\}$, this k is necessarily positive. We now prove, by induction on i, that $\mathcal{C}^i(\mathfrak{g}) \subset \mathcal{C}_{k-i}$. When $i = 0$, this inclusion is just $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g} = \mathcal{C}_k$, which is already true. Now assume that for $i \geq 0$, $\mathcal{C}^i(\mathfrak{g}) \subset \mathcal{C}_{k-i}$. Then $\mathcal{C}^{i+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^i(\mathfrak{g})] \subset [\mathfrak{g}, \mathcal{C}_{k-i}] \subset \mathcal{C}_{k-i-1}$, the last inclusion arising from the condition that $\mathcal{C}_{k-i} \subset \mathfrak{c}(\mathfrak{g}/\mathcal{C}_{k-i-1})$. This completes the induction.

When $i = k$, we therefore obtain $\mathcal{C}^k(\mathfrak{g}) \subset \mathcal{C}_0 = \{0\}$. Hence $\mathfrak{g}$ is nilpotent. □

Exercise 5.1.5. Show that $\mathfrak{g}$ is nilpotent if and only if it has a central series which reaches $\{0\}$; that is, there is a descending sequence of ideals of $\mathfrak{g}$:

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\}$$

where $\mathfrak{g}_i/\mathfrak{g}_{i+1} \subset \mathfrak{c}(\mathfrak{g}/\mathfrak{g}_{i+1})$.

Exercise 5.1.6. Prove that if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}$ is solvable.

Example 5.1.7. Recall that $\mathfrak{g} = T_n(\mathbb{F})$ is the solvable Lie algebra of $n \times n$ upper triangular matrices over $\mathbb{F}$. Then $T_n(\mathbb{F}) = D_n(\mathbb{F}) \oplus U_n(\mathbb{F})$, where $D_n(\mathbb{F})$ is the vector space of diagonal $n \times n$ matrices and $U_n(\mathbb{F})$ is the vector space of strictly upper triangular $n \times n$ matrices. We saw in Example 4.1.9 that $\mathfrak{g}' \subset U_n(\mathbb{F})$.

In fact $\mathfrak{g}' = U_n(\mathbb{F})$, since $U_n(\mathbb{F})$ has basis $\{E_{ij}\}_{i<j}$, and

$$E_{ij} = [E_{ii}, E_{ij}].$$

The above equation also shows that $[D_n(\mathbb{F}), U_n(\mathbb{F})] = U_n(\mathbb{F})$. Hence $\mathcal{C}^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}'] \supset [D_n(\mathbb{F}), U_n(\mathbb{F})] = U_n(\mathbb{F}) = \mathfrak{g}'$, so $\mathcal{C}^2(\mathfrak{g}) = \mathfrak{g}'$, and it follows that $\mathcal{C}^i(\mathfrak{g}) = \mathfrak{g}'$ for all $i \geq 1$. Thus $T_n(\mathbb{F})$ is not nilpotent.

On the other hand, the Lie algebra $\mathfrak{h} = U_n(\mathbb{F})$ is nilpotent. Using the notation of Example 4.1.9, let $\mathfrak{g}_r$ denote the subspace of $T_n(\mathbb{F})$ consisting of those matrices with 0's below the diagonal r steps above the main diagonal. We claim, using induction on r, that $\mathcal{C}^r(\mathfrak{h}) = \mathfrak{g}_r$ for all $r \geq 1$. For $r = 1$, this is just the equality $\mathfrak{g}_1 = U_n(\mathbb{F}) = \mathfrak{h}$. Assuming that the claim is true for r, the corresponding equality for $r + 1$ will follow if we can show that $[\mathfrak{g}_1, \mathfrak{g}_r] = \mathfrak{g}_{r+1}$. Now $\mathfrak{g}_r$ is spanned by the elementary matrices E_{kl}, where $l \geq k + r$. Suppose that $E_{ij} \in \mathfrak{g}_1$ and $E_{kl} \in \mathfrak{g}_r$. Then, equation (4.4) says that

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

If $j = k$, then $l \geq k + r = j + r \geq i + 1 + r$, so $E_{il} \in \mathfrak{g}_{r+1}$. If $l = i$, then $j \geq i + 1 = l + 1 \geq k + r + 1$, so $E_{kj} \in \mathfrak{g}_{r+1}$. Either way, the Lie bracket above belongs to $\mathfrak{g}_{r+1}$, and this shows that $[\mathfrak{g}_1, \mathfrak{g}_r] \subset \mathfrak{g}_{r+1}$.

On the other hand, if $l \geq k+r+1$, then

$$E_{kl} = E_{k,k+1}\, E_{k+1,l} = [E_{k,k+1}, E_{k+1,l}] \in [\mathfrak{g}_1, \mathfrak{g}_r],$$

so $\mathfrak{g}_{r+1} \subset [\mathfrak{g}_1, \mathfrak{g}_r]$. We have thus proved that $[\mathfrak{g}_1, \mathfrak{g}_r] = \mathfrak{g}_{r+1}$.

Of course, when $r = n$, we get $\mathcal{C}^n(\mathfrak{h}) = \mathfrak{g}_{n+1} = \{0\}$. Thus $\mathfrak{h} = U_n(\mathbb{F})$ is nilpotent.

Exercise 5.1.8. Prove that the Lie algebras $T_n(\mathbb{F})$ and $U_n(\mathbb{F})$ both have centers of dimension 1.

Exercise 5.1.9. (See the book by Binz and Pods [BP08]) The *general Heisenberg Lie algebra* $\mathfrak{h}_n$ is defined as the space of $(n+2)\times(n+2)$ matrices over $\mathbb{F}$ which have block form

$$\begin{pmatrix} 0 & {}^t v & c \\ 0 & \mathbf{0}_n & w \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

where v, $w \in \mathbb{F}^n$, $c \in \mathbb{F}$, and $\mathbf{0}_n$ is the zero $n \times n$ matrix. Prove that $\mathfrak{h}_n$ is a nilpotent Lie algebra of dimension $2n+1$. What is the minimum k such that $\mathcal{C}^k(\mathfrak{h}_n) = \{0\}$? (The algebra $\mathfrak{h}_n$ is the Lie algebra of the *Heisenberg group*, which in the case $\mathbb{F} = \mathbb{R}$ is used in the description of n-dimension quantum mechanical systems.)

Exercise 5.1.10. Show that any non-abelian two-dimensional Lie algebra over $\mathbb{F}$ is solvable, but not nilpotent. (See Exercise 3.2.2.)

Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homorphism. Then the image $\varphi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{h}$, and it is clear that $\varphi(\mathcal{C}^i(\mathfrak{g})) = \mathcal{C}^i(\varphi(\mathfrak{g}))$, for all i. Thus, if $\mathfrak{g}$ is nilpotent, then so are all homomorphic images of $\mathfrak{g}$.

Exercise 5.1.11. Suppose that $\mathfrak{g}$ is nilpotent. Prove that:

1. All subalgebras of $\mathfrak{g}$ are nilpotent.
2. If $\mathfrak{g} \neq \{0\}$, then its center $\mathfrak{c} \neq \{0\}$.

Proposition 5.1.12. *Let $\mathfrak{g}$ be a Lie algebra, with center $\mathfrak{c}$. Then $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}/\mathfrak{c}$ is nilpotent.*

Proof. Suppose that $\mathfrak{g}$ is nilpotent. Then $\mathfrak{g}/\mathfrak{c}$ is the homomorphic image of $\mathfrak{g}$ under the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}$. Thus $\mathfrak{g}/\mathfrak{c}$ is nilpotent.

Conversely, suppose that $\mathfrak{g}/\mathfrak{c}$ is nilpotent. If $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}$ is the natural projection, then $\pi(\mathcal{C}^i(\mathfrak{g})) = \mathcal{C}^i(\mathfrak{g}/\mathfrak{c})$. By hypothesis, $\mathcal{C}^k(\mathfrak{g}/\mathfrak{c}) = \{\mathfrak{c}\}$ (the zero subspace in $\mathfrak{g}/\mathfrak{c}$) for some k. Thus, $\mathcal{C}^k(\mathfrak{g}) \subset \mathfrak{c}$, from which we conclude that $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^k(\mathfrak{g})] \subset [\mathfrak{g}, \mathfrak{c}] = \{0\}$. $\square$

Exercise 5.1.13. Prove or give a counterexample: suppose that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. If $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are nilpotent, then $\mathfrak{g}$ is nilpotent. (See Proposition 4.1.10.)

5.2 Engel's Theorem

In this section, our objective is is to prove the following important result.

Theorem 5.2.1. *(Engel's Theorem) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}$ is nilpotent if and only if, for all $x \in \mathfrak{g}$, $\operatorname{ad} x$ is a nilpotent linear operator on $\mathfrak{g}$.*

It is easy to prove that if $\mathfrak{g}$ is a nilpotent Lie algebra, then $\operatorname{ad} x$ is a nilpotent linear transformation, for all $x \in \mathfrak{g}$. Indeed, for any $y \in \mathfrak{g}$ and $k \geq 1$, we have

$$(\operatorname{ad} x)^k(y) = \underbrace{[x, [x, [\cdots, [x}_{k \text{ times}}, y]]]] \in \mathcal{C}^k(\mathfrak{g}).$$

Thus, if $\mathcal{C}^k(\mathfrak{g}) = \{0\}$, then $(\operatorname{ad} x)^k = 0$ for all $x \in \mathfrak{g}$.

To prove the opposite implication, we make use of the following lemma, which like Dynkin's lemma (Lemma 4.2.1), asserts the existence of a common eigenvector.

Lemma 5.2.2. *(Engel) Let V be a vector space over $\mathbb{F}$. Suppose that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent linear operators on V. Then there exists a vector $v \neq 0$ in V such that $X(v) = 0$ for all $X \in \mathfrak{g}$.*

Proof. We prove this lemma by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 0$, there is nothing to prove.

Therefore, we may assume that $n \geq 1$ and that the lemma holds for all Lie subalgebras, of dimension $< n$, of all $\mathfrak{gl}(W)$, for all vector spaces W over $\mathbb{F}$. Then, let $\mathfrak{g}$ be an n-dimensional Lie subalgebra of $\mathfrak{gl}(V)$, for some vector space V, such that the elements of $\mathfrak{g}$ are all nilpotent linear maps on V. The induction proceeds along several steps:

Step 1: $\mathfrak{g}$ acts on itself via the adjoint representation. We claim that $\operatorname{ad} X$ is a nilpotent linear map on $\mathfrak{g}$, for each $X \in \mathfrak{g}$.

Proof of Step 1: Since $\mathfrak{g} \subset \mathfrak{gl}(V)$, the Lie bracket in $\mathfrak{g}$ is the commutator product: $[X, Y] = XY - YX$. For $X \in \mathcal{L}(V)$, denote by L_X and R_X the operators defined on $\mathcal{L}(V)$ by multiplication by X on the left and on the right respectively:

$$L_X : Y \longmapsto XY \qquad , \qquad R_X : Y \longmapsto YX,$$

so that

$$\operatorname{ad} X = L_X - R_X.$$

Notice that L_X and R_X commute as linear maps on $\mathcal{L}(V)$:

$$L_X \circ R_X(Y) = XYX = R_X \circ L_X(Y).$$

Hence, by the binomial formula,

$$\begin{aligned}(\operatorname{ad} X)^m (Y) &= (L_X - R_X)^m(Y)\\ &= \sum_{j=0}^{m} \binom{m}{j} L_X^j \circ (-R_X)^{m-j} (Y)\\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} X^j\, Y\, X^{m-j},\end{aligned}$$

for all X, $Y \in \mathfrak{g}$. Now each $X \in \mathfrak{g}$ is a nilpotent linear map on V, so $X^k = 0$ for some k. If we let $m = 2k$ in the above equation, we see that $(\operatorname{ad} X)^m(Y) = 0$ for all $Y \in \mathfrak{g}$. Hence $\operatorname{ad} X$ is nilpotent.

Step 2: Let $\mathfrak{m}$ be a maximal proper subalgebra of $\mathfrak{g}$. Then there exists an $X_0 \in \mathfrak{g} \setminus \mathfrak{m}$ such that $[X_0, \mathfrak{m}] \subset \mathfrak{m}$.

Proof of Step 2: Recall that $\mathfrak{m}$ *proper* means that $\mathfrak{m} \subsetneq \mathfrak{g}$ (however, $\mathfrak{m}$ could very well be $\{0\}$). The algebra $\mathfrak{m}$ acts on the *vector space* $\mathfrak{g}/\mathfrak{m}$ *via* ad, that is, for each $Z \in \mathfrak{m}$, define the map $\widetilde{\operatorname{ad}}\, Z$ on $\mathfrak{g}/\mathfrak{m}$ by

$$\widetilde{\operatorname{ad}}\, Z\,(Y + \mathfrak{m}) = \operatorname{ad} Z\,(Y) + \mathfrak{m}.$$

It is easy to check that $\widetilde{\operatorname{ad}}\, Z$ is a well-defined linear map on $\mathfrak{g}/\mathfrak{m}$. For each $Z \in \mathfrak{m}$, it follows from *Step 1* that $\widetilde{\operatorname{ad}}\, Z$ is a nilpotent linear map, since

$$\left(\widetilde{\operatorname{ad}}\, Z\right)^m (Y + \mathfrak{m}) = (\operatorname{ad} Z)^m\ (Y) + \mathfrak{m} = \mathfrak{m},$$

for sufficiently large m.

Moreover, the map $Z \mapsto \widetilde{\operatorname{ad}}\, Z$ is easily seen to be a Lie algebra homomorphism from $\mathfrak{m}$ into $\mathfrak{gl}\,(\mathfrak{g}/\mathfrak{m})$. (It suffices to show that $\widetilde{\operatorname{ad}}\,[Z_1, Z_2] = [\widetilde{\operatorname{ad}}\, Z_1, \widetilde{\operatorname{ad}}\, Z_2]$ for Z_1, $Z_2 \in \mathfrak{m}$, which follows immediately from the same relation for ad.) Thus $\widetilde{\operatorname{ad}}\,\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{gl}\,(\mathfrak{g}/\mathfrak{m})$ consisting of nilpotent linear maps on $\mathfrak{g}/\mathfrak{m}$.

Since $\dim(\widetilde{\operatorname{ad}}\,\mathfrak{m}) \leq \dim \mathfrak{m} < \dim \mathfrak{g}$, the induction hypothesis says that there exists a nonzero element $X_0 + \mathfrak{m} \in \mathfrak{g}/\mathfrak{m}$ such that $\widetilde{\operatorname{ad}}\, Z(X_0 + \mathfrak{m}) = \mathfrak{m}$ for all $Z \in \mathfrak{m}$. This means that the representative $X_0 \in \mathfrak{g}$ satisfies $X_0 \notin \mathfrak{m}$ and $\operatorname{ad} Z\,(X_0) \in \mathfrak{m}$ for all $Z \in \mathfrak{m}$.

Step 3: $\mathfrak{m} + \mathbb{F}X_0 = \mathfrak{g}$, and $\mathfrak{m}$ is an ideal of $\mathfrak{g}$.

Proof of Step 3: The sum $\mathfrak{m} + \mathbb{F}X_0$ is a subalgebra of $\mathfrak{g}$, since

$$[\mathfrak{m} + \mathbb{F}X_0, \mathfrak{m} + \mathbb{F}X_0] \subset [\mathfrak{m}, \mathfrak{m}] + [\mathfrak{m}, \mathbb{F}X_0] + [\mathbb{F}X_0, \mathfrak{m}] \subset \mathfrak{m}. \tag{5.2}$$

Since $X_0 \notin \mathfrak{m}$, $\mathfrak{m} + \mathbb{F}X_0$ is a subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{m}$. But $\mathfrak{m}$ is a maximal proper subalgebra of $\mathfrak{g}$; thus $\mathfrak{g} = \mathfrak{m} + \mathbb{F}X_0$. Equation (5.2) above shows that $\mathfrak{m}$ is an ideal of $\mathfrak{g}$.

Step 4: There exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

Proof of Step 4: Now $\mathfrak{m}$ is a subalgebra of $\mathfrak{gl}(V)$ whose elements are all nilpotent. Since $\dim \mathfrak{m} < \dim \mathfrak{g}$, we can again apply the induction hypothesis to conclude that

$$W := \{v \in V \mid Z(v) = 0 \text{ for all } Z \in \mathfrak{m}\} \neq \{0\}.$$

W is a joint eigenspace of $\mathfrak{m}$, corresponding to the zero linear functional, so is a subspace of V. Moreover W is X_0-invariant: in fact, for any $w \in W$ and $Z \in \mathfrak{m}$, we have

$$\begin{aligned} Z(X_0\,(w)) &= (ZX_0 - X_0Z)(w) + X_0Z\,(w) \\ &= [Z, X_0](w) + X_0(0) \\ &= 0, \end{aligned}$$

since $[Z, X_0] \in \mathfrak{m}$. Thus $X_0(w) \in W$, and W is X_0-invariant. Now the restriction $X_0|_W$ is a nilpotent linear map on W, so X_0 annihilates a nonzero vector $w_0 \in W$: $X_0(w_0) = 0$. Since $\mathfrak{m}$ also annihilates w_0 and $\mathfrak{g} = \mathfrak{m} + \mathbb{F}X_0$, we see that $Y(w_0) = 0$ for all $Y \in \mathfrak{g}$. Putting $v = w_0$, our lemma is proved. □

We are now ready to finish the proof of Engel's Theorem.

Proof of Engel's Theorem. It suffices to prove that if $\mathfrak{g}$ is a Lie algebra such that $\operatorname{ad} x$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent. We will do this by induction on $\dim \mathfrak{g}$, the cases $\dim \mathfrak{g} = 0$ and $\dim \mathfrak{g} = 1$ being trivial.

So assume that the result given above holds for all Lie algebras of dimension $< n$, and that $\dim \mathfrak{g} = n$, with $\operatorname{ad} x$ nilpotent for all $x \in \mathfrak{g}$.

Then $\operatorname{ad} \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ consisting of nilpotent linear maps, so by Lemma 5.2.2, there exists a nonzero element $z \in \mathfrak{g}$ such that $\operatorname{ad} x\,(z) = 0$ for all $x \in \mathfrak{g}$. This implies that z lies in the center $\mathfrak{c}$ of $\mathfrak{g}$, so $\mathfrak{c} \neq \{0\}$.

Consider the quotient algebra $\mathfrak{g}/\mathfrak{c}$. Let $\operatorname{ad}_{\mathfrak{c}}$ denote its adjoint representation. Then

$$\operatorname{ad}_{\mathfrak{c}}(x + \mathfrak{c})\,(y + \mathfrak{c}) = [x + \mathfrak{c}, y + \mathfrak{c}] = [x, y] + \mathfrak{c}$$

For each $x \in \mathfrak{g}$, $\operatorname{ad}_{\mathfrak{c}}(x + \mathfrak{c})$ is a nilpotent linear map on $\mathfrak{g}/\mathfrak{c}$. Thus, by the induction hypothesis (since $\dim(\mathfrak{g}/\mathfrak{c}) < \dim \mathfrak{g}$), the Lie algebra $\mathfrak{g}/\mathfrak{c}$ is nilpotent. Then by Proposition 5.1.12, $\mathfrak{g}$ is nilpotent.

This completes the induction step and the proof of Engel's Theorem.

The following theorem is the nilpotent analogue of Lie's Theorem (Theorem 4.2.3). While Lie's Theorem only holds for complex vector spaces, the theorem below holds for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Theorem 5.2.3. *(Engel's Structure Theorem) Let $\mathfrak{g}$ be a Lie algebra consisting of nilpotent linear maps acting on a vector space V. Then there is a basis of V relative to which the matrix of every element of $\mathfrak{g}$ is strictly upper triangular.*

Proof. The proof is by induction on $\dim V$. If $\dim V = 0$ or $\dim V = 1$, this result is trivial, since any nilpotent linear map on V is just 0.

Assume that the result is true for dimension $n-1$, and let V have dimension n. Now by Lemma 5.2.2, there exists a nonzero vector v_1 such that $X(v_1) = 0$ for all $X \in \mathfrak{g}$. Let $V_1 = \mathbb{F}v_1$. Since V_1 is $\mathfrak{g}$-invariant, $\mathfrak{g}$ acts on the quotient space V/V_1 via

$$X \cdot (v + V_1) = X(v) + V_1$$

for all $v \in V$ and $X \in \mathfrak{g}$. Each $X \in \mathfrak{g}$ is clearly a nilpotent linear map on V/V_1. Hence V/V_1 has a basis $(v_2 + V_1, \ldots, v_n + V_1)$ relative to which the matrix of each $X \in \mathfrak{g}$ is strictly upper triangular. Then $(v_1, v_2, \ldots, v_n)$ is a basis of V, and it is also clear that the matrix of each $X \in \mathfrak{g}$ with respect to this basis is strictly upper triangular. □

Theorem 5.2.4. *Let $\mathfrak{g}$ be a solvable Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. By Lie's Theorem (Theorem 4.2.5), there is a basis of $\mathfrak{g}$ with respect to which the matrix of $\operatorname{ad} x$ is upper triangular, for each $x \in \mathfrak{g}$. Thus, for any x, $y \in \mathfrak{g}$, the matrix of $\operatorname{ad}[x, y] = [\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x$ with respect to this basis is strictly upper triangular. Since strictly upper triangular matrices correspond to nilpotent linear maps, it follows that $\operatorname{ad} w$ is nilpotent for each $w \in \mathfrak{g}'$. Therefore the restriction $\operatorname{ad} w|_{\mathfrak{g}'}$ is also nilpotent. By Engel's Theorem, this implies that $\mathfrak{g}'$ is nilpotent. □

The converse holds for any field $\mathbb{F}$.

Theorem 5.2.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ such that $\mathfrak{g}'$ is nilpotent. Then $\mathfrak{g}$ is solvable.*

Proof. Since $\mathfrak{g}'$ is nilpotent, by Exercise 5.1.6 it is solvable. Moreover, the quotient Lie algebra $\mathfrak{g}/\mathfrak{g}'$ is abelian, hence solvable. Thus, by Proposition 4.1.10, $\mathfrak{g}$ is solvable. □

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