Chapter 4

Solvable Lie Algebras and Lie's Theorem

4.1 Solvable Lie Algebras

Definition 4.1.1. The *derived algebra* of a Lie algebra $\mathfrak{g}$ is $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

$\mathfrak{g}'$ is an ideal of $\mathfrak{g}$, since $\mathfrak{g}'$ is spanned by the products $[x, y]$, for all x, $y \in \mathfrak{g}$, and $[[x, y], z] \in [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$, for all x, y, $z \in \mathfrak{g}$.

We could abbreviate the argument that $\mathfrak{g}'$ is an ideal by writing $[\mathfrak{g}', \mathfrak{g}] = [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$.

Theorem 4.1.2. *$\mathfrak{g}/\mathfrak{g}'$ is abelian, and for any ideal $\mathfrak{a}$, $\mathfrak{g}/\mathfrak{a}$ is abelian $\iff$ $\mathfrak{g}' \subset \mathfrak{a}$.*

Proof. For any x, $y \in \mathfrak{g}$, we have $[x + \mathfrak{g}', y + \mathfrak{g}'] = [x, y] + \mathfrak{g}' = \mathfrak{g}'$, so $\mathfrak{g}/\mathfrak{g}'$ is abelian.

Let us now prove the second assertion. Now

$$\begin{aligned} \mathfrak{g}/\mathfrak{a} \text{ is abelian} &\iff [x + \mathfrak{a}, y + \mathfrak{a}] = [x, y] + \mathfrak{a} = \mathfrak{a} \text{ for all } x, y \in \mathfrak{g} \\ &\iff [x, y] \in \mathfrak{a} \text{ for all } x, y \in \mathfrak{g} \\ &\iff \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a} \end{aligned}$$

□

Definition 4.1.3. A *characteristic ideal* of $\mathfrak{g}$ is an ideal $\mathfrak{a}$ such that $D(\mathfrak{a}) \subset \mathfrak{a}$ for every derivation D of $\mathfrak{g}$.

The derived ideal $\mathfrak{g}'$ is a characteristic ideal: for every derivation D, we have $D(\mathfrak{g}') = D[\mathfrak{g}, \mathfrak{g}] = [D(\mathfrak{g}), \mathfrak{g}] + [\mathfrak{g}, D(\mathfrak{g})] \subset [\mathfrak{g}, \mathfrak{g}] + [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$.

Proposition 4.1.4. *Let $\mathfrak{h}$ be any vector subspace of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}' \subset \mathfrak{h}$. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. We have $[\mathfrak{g}, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}' \subset \mathfrak{h}$. □

Let $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = \mathfrak{g}'$, and for any $i \geq 2$, define $\mathfrak{g}^{(i)}$ inductively by $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$.

Definition 4.1.5. We call $\mathfrak{g}^{(i)}$ the *ith derived algebra of* $\mathfrak{g}$. *The derived series of* $\mathfrak{g}$ *is*

$$\mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(i)} \supset \mathfrak{g}^{(i+1)} \supset \cdots \tag{4.1}$$

Proposition 4.1.6. *The derived series consists of a decreasing sequence of characteristic ideals of* $\mathfrak{g}$.

Proof. We need to prove that each $\mathfrak{g}^{(i)}$ is a characteristic ideal of $\mathfrak{g}$. This is done by induction on i, noting that there is nothing to prove for $i = 0$, and that we have already proved that $\mathfrak{g}^{(1)} = \mathfrak{g}'$ is a characteristic ideal in the remark before Definition 4.1.5. So assume that $\mathfrak{g}^{(i)}$ is a characteristic ideal of $\mathfrak{g}$. Then by the Jacobi identity and the induction hypothesis,

$$[\mathfrak{g}, \mathfrak{g}^{(i+1)}] = [\mathfrak{g}, [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]] \subset [\mathfrak{g}^{(i)}, [\mathfrak{g}, \mathfrak{g}^{(i)}]] + [\mathfrak{g}^{(i)}, [\mathfrak{g}^{(i)}, \mathfrak{g}]] \subset [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] = \mathfrak{g}^{(i+1)}.$$

It follows that $\mathfrak{g}^{(i+1)}$ is an ideal of $\mathfrak{g}$. Next, for any $D \in \operatorname{Der}\mathfrak{g}$, we have

$$\begin{aligned} D(\mathfrak{g}^{(i+1)}) &= D[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \\ &= [D(\mathfrak{g}^{(i)}), \mathfrak{g}^{(i)}] + [\mathfrak{g}^{(i)}, D(\mathfrak{g}^{(i)})] \\ &\subset [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \qquad \text{(by the induction hypothesis)} \\ &= \mathfrak{g}^{(i+1)}. \end{aligned}$$

□

Definition 4.1.7. $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^{(k)} = \{0\}$ for some $k \in \mathbb{Z}^+$.

Exercise 4.1.8. Show that $\mathfrak{g}$ is solvable if and only if there is a nested sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\}$ such that $\mathfrak{g}_{i+1}$ is an ideal of $\mathfrak{g}_i$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Note that no simple Lie algebra can be solvable. In fact, if $\mathfrak{g}$ is simple, then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is a nonzero ideal of $\mathfrak{g}$ (since $\mathfrak{g}$ is, by definition, non-abelian); hence $\mathfrak{g}' = \mathfrak{g}$. Thus $\mathfrak{g}'' = \mathfrak{g}' = \mathfrak{g}$, etc, and the derived series is constant. In particular, $\mathfrak{sl}(2, \mathbb{C})$ is not solvable.

Example 4.1.9. Let $\mathfrak{g} = T_n(\mathbb{F})$ be the vector space of upper triangular $n \times n$ matrices over $\mathbb{F}$. If A and B are upper triangular matrices

$$A = \begin{pmatrix} s_1 & & & * \\ 0 & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{pmatrix}, \qquad B = \begin{pmatrix} t_1 & & & * \\ 0 & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix}$$

then the product AB has the form

$$AB = \begin{pmatrix} s_1 t_1 & & & * \\ 0 & s_2 t_2 & & \\ & & \ddots & \\ 0 & & & s_n t_n \end{pmatrix}$$

and likewise, BA has the same form. Hence the commutator $AB - BA$ is strictly upper triangular

$$AB - BA = \begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \tag{4.2}$$

Thus the elements of $\mathfrak{g}'$ consist of strictly upper triangular matrices. With a bit of thought, one can see that the elements of $\mathfrak{g}^{(2)}$ are matrices whose entries are 0's below the diagonal 2 steps above the main diagonal; that is, $\mathfrak{g}^{(2)}$ consists of matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i \geq j - 1$.

$$\begin{pmatrix} 0 & 0 & * & & * & * \\ & 0 & 0 & * & & * \\ & & 0 & 0 & & \\ & & & \ddots & \ddots & * \\ & & & & 0 & 0 \\ 0 & & & & & 0 \end{pmatrix}$$

The $\mathfrak{g}^{(3)}$ matrices have 0's below the diagonal 2^2 steps above the main diagonal. Generally, $\mathfrak{g}^{(i)}$ matrices have 0's below the diagonal 2^{i-1} steps above the main diagonal.

We can also use Exercise 4.1.8 to show that $T_n(\mathbb{F})$ is solvable. First, for any i, j, let E_{ij} be the $n \times n$ matrix whose (i, j) entry is a 1 and all of whose other entries are 0. Then $\{E_{ij}\}_{1\le i,j\le n}$ is a basis of $\mathfrak{gl}(n,\mathbb{F})$. The E_{ij} satisfy the multiplication rules

$$E_{ij}E_{kl} = \delta_{jk}E_{il}, \tag{4.3}$$

and so

$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{li}E_{kj} \tag{4.4}$$

Now

$$T_n(\mathbb{F}) = \bigoplus_{i\le j}(\mathbb{F}E_{ij}).$$

For each integer $r \ge 0$, let $\mathfrak{g}_r$ denote the subspace of $T_n(\mathbb{F})$ consisting of those matrices whose entries below the diagonal r steps above the main diagonal are 0. Then

$$\mathfrak{g}_r = \bigoplus_{k\le l-r}(\mathbb{F}E_{kl})$$

Note that $T_n(\mathbb{F}) = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n \supset \mathfrak{g}_{n+1} = \{0\}$. We claim that $\mathfrak{g}_r$ is an ideal of $T_n(\mathbb{F})$ and that $\mathfrak{g}_r/\mathfrak{g}_{r+1}$ is abelian.

To prove that $\mathfrak{g}_r$ is an ideal of $T_n(\mathbb{F})$, we just need to prove that $[E_{ij}, E_{kl}] \in \mathfrak{g}_r$ whenever $E_{ij} \in T_n(\mathbb{F})$ and $E_{kl} \in \mathfrak{g}_r$. For this, we apply the commutation rule (4.4). The right hand side of (4.4) is nonzero only if $j = k$ or $l = i$. If $j = k$, then $i \le j = k \le l - r$, so $E_{il} \in \mathfrak{g}_r$. If $l = i$, then $k \le l - r = i - r \le j - r$, so $E_{kj} \in \mathfrak{g}_r$. Either way, we get $[E_{ij}, E_{kl}] \in \mathfrak{g}_r$.

The condition that $\mathfrak{g}_r/\mathfrak{g}_{r+1}$ is abelian is the same as the condition that $[\mathfrak{g}_r, \mathfrak{g}_r] \subset \mathfrak{g}_{r+1}$. For $r = 0$, the proof is the same as the argument leading up to equation (4.2).

For $r \ge 1$, we will show that if E_{ij} and E_{kl} are in $\mathfrak{g}_r$, then $[E_{ij}, E_{kl}] \in \mathfrak{g}_{r+1}$. For this, it suffices, in turn, to show that the matrix product $E_{ij}E_{kl}$ lies in $\mathfrak{g}_{r+1}$. (The argument that $E_{kl}E_{ij} \in \mathfrak{g}_{r+1}$ is the same.)

Now, by (4.3), $E_{ij}E_{kl}$ is nonzero if and only if $j = k$, in which case the product is E_{il}. But this means that $i \le j - r = k - r \le l - 2r \le l - (r + 1)$, since $r \ge 1$. Thus $E_{il} \in \mathfrak{g}_{r+1}$.

We have thus shown that for all $r \ge 0$, $[\mathfrak{g}_r, \mathfrak{g}_r] \subset \mathfrak{g}_{r+1}$, and hence $\mathfrak{g} = T_n(\mathbb{F})$ is solvable. □

We now make the the following observations about solvable Lie algebras. First, if $\mathfrak{g}$ is solvable, then so is any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This is because if $\mathfrak{h}^{(i)}$ is the ith term in the derived series for $\mathfrak{h}$, then a simple induction argument shows that $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$ for all i. The second observation is that if $\mathfrak{g}$ is solvable, then so is any homomorphic image of $\mathfrak{g}$. In fact, suppose that $\varphi : \mathfrak{g} \to \mathfrak{m}$ is a Lie algebra homomorphism. Then the image $\mathfrak{q} = \varphi(\mathfrak{g})$ is a subalgebra of $\mathfrak{m}$, and it is easy to see, using another simple induction argument, that $\varphi(\mathfrak{g}^{(i)}) = \mathfrak{q}^{(i)}$ for all i.

Proposition 4.1.10. *If* $\mathfrak{g}$ *is a Lie algebra and* $\mathfrak{a}$ *is an ideal of* $\mathfrak{g}$*, then* $\mathfrak{g}$ *is solvable* $\iff$ *both* $\mathfrak{a}$ *and* $\mathfrak{g}/\mathfrak{a}$ *are solvable.*

Proof. If $\mathfrak{g}$ is solvable, then so is $\mathfrak{a}$, since the ideal $\mathfrak{a}$ is also a subalgebra of $\mathfrak{g}$. The quotient algebra $\mathfrak{g}/\mathfrak{a}$ is the homomorphic image of $\mathfrak{g}$ under the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$, so it must also be solvable.

Conversely, suppose that both $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable. Since $\mathfrak{g}/\mathfrak{a}$ is solvable, we must have $(\mathfrak{g}/\mathfrak{a})^{(k)} = \{0\}$, for some k. (The "0" here refers to the zero vector in $\mathfrak{g}/\mathfrak{a}$.) But $(\mathfrak{g}/\mathfrak{a})^{(k)} = \pi(\mathfrak{g})^{(k)} = \pi((\mathfrak{g})^{(k)})$. It follows that $\mathfrak{g}^{(k)} \subset \mathfrak{a}$, and from this, it follows that $\mathfrak{g}^{(k+r)} \subset \mathfrak{a}^{(r)}$, for all r. But then $\mathfrak{a}$ is solvable, so $\mathfrak{a}^{(m)} = \{0\}$ for some m, whence $\mathfrak{g}^{(k+m)} = \{0\}$. Therefore, $\mathfrak{g}$ is solvable. □

Corollary 4.1.11. *Suppose that* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are solvable ideals of any Lie algebra* $\mathfrak{g}$*. Then* $\mathfrak{a} + \mathfrak{b}$ *is a solvable ideal of* $\mathfrak{g}$*.*

Proof. It is easy to see that $\mathfrak{a} + \mathfrak{b}$ is an ideal of $\mathfrak{g}$. Now $\mathfrak{b}$ is also an ideal of $\mathfrak{a} + \mathfrak{b}$, and by the Isomorphism Theorem (Theorem 3.2.7), $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$. But the quotient algebra $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ is solvable by the preceding proposition. Hence both $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ and $\mathfrak{b}$ are solvable, so again by the preceding proposition, we see that $\mathfrak{a} + \mathfrak{b}$ is solvable. □

Theorem 4.1.12. *Any (finite-dimensional) Lie algebra* $\mathfrak{g}$ *has a maximal solvable ideal* $\mathfrak{R}_s$*, which contains every solvable ideal of* $\mathfrak{g}$*.*

Proof. Since $\{0\}$ is a solvable ideal of $\mathfrak{g}$, the collection of all solvable ideals of $\mathfrak{g}$ is nonempty. In this collection, let $\mathfrak{R}_s$ be a solvable ideal of *maximal* dimension. If $\mathfrak{a}$ is any solvable ideal, then by Corollary 4.1.11, $\mathfrak{R}_s + \mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, whence by the maximality of $\mathfrak{R}_s$, we conclude that $\mathfrak{R}_s + \mathfrak{a} = \mathfrak{R}_s$, and so $\mathfrak{a} \subset \mathfrak{R}_s$. □

Definition 4.1.13. $\mathfrak{R}_s$ is called the *solvable radical* of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is said to be *semisimple* if $\mathfrak{R}_s = \{0\}$.

Corollary 4.1.14. *If* $\mathfrak{g}$ *is simple, then* $\mathfrak{g}$ *is semisimple.*

Proof. We had previously observed that since $\mathfrak{g}$ is simple, the derived series for $\mathfrak{g}$ is constant: $\mathfrak{g}^{(i)} = \mathfrak{g}$ for all i. Thus, $\mathfrak{g} \neq \mathfrak{R}_s$, but $\mathfrak{R}_s$ is an ideal of $\mathfrak{g}$, so this forces $\mathfrak{R}_s = \{0\}$. □

Are there semisimple Lie algebras which are not simple? Sure! For an example, we first introduce the notion of an external direct sum of Lie algebras.

Let V and W be vector spaces over $\mathbb{F}$. The Cartesian product $V \times W$ has the structure of a vector space, where addition and scalar multiplication are defined by

$$\begin{aligned}(v_1, w_1) + (v_2, w_2) &= (v_1 + v_2, w_1 + w_2)\\ \alpha\,(v, w) &= (\alpha v, \alpha w),\end{aligned}$$

for all $v, v_1, v_2 \in V$, all $w, w_1, w_2 \in W$, and all $\alpha \in \mathbb{F}$. Equipped with this vector space structure, we call $V_1 \times V_2$ the *external direct sum* of V_1 and V_2.

The external direct sum $V_1 \times V_2 \times \cdots \times V_k$ of k vector spaces is defined similarly.

Exercise 4.1.15. (Easy)

Exercise 4.1.16. (Straightforward)

Exercise 4.1.17.

Corollary 4.1.18. *If $\mathfrak{g}$ is any Lie algebra and $\mathfrak{R}_s$ is its solvable radical, then the quotient algebra $\mathfrak{g}/\mathfrak{R}_s$ is semisimple.*

Proof. Let $\mathfrak{I}$ denote the solvable radical of $\mathfrak{g}/\mathfrak{R}_s$. Then, by the Correspondence Theorem (Theorem 3.2.6), we have $\mathfrak{I} = \overline{\mathfrak{R}}/\mathfrak{R}_s$, where $\overline{\mathfrak{R}}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{R}_s$. But since both $\overline{\mathfrak{R}}/\mathfrak{R}_s$ and $\mathfrak{R}_s$ are solvable, it follows from Corollary 4.1.11 that $\overline{\mathfrak{R}}$ is solvable. Since $\mathfrak{R}_s$ is maximal solvable, we conclude that $\overline{\mathfrak{R}} = \mathfrak{R}_s$, so $\mathfrak{I} = \{0\}$. This shows that $\mathfrak{g}/\mathfrak{R}_s$ is semisimple. □

Exercise 4.1.19. Suppose that $\mathfrak{g}$ is solvable. Show that $\mathfrak{g}$ has no semisimple subalgebra $\neq \{0\}$.

4.2 Lie's Theorem

Let V be a nonzero vector space over $\mathbb{F}$. Let us recall that $\mathfrak{gl}(V)$ is the Lie algebra of all linear operators on V (same as $\mathcal{L}(V)$), in which the Lie bracket is the commutator $[A, B] = AB - BA$. If we fix a basis B of V, then the map which takes any $T \in \mathfrak{gl}(V)$ into its matrix $M(T)$ with respect to B is a Lie algebra isomorphism from $\mathfrak{gl}(V)$ onto $\mathfrak{gl}(n, \mathbb{F})$.

Our objective now is to prove Lie's Theorem, which says that, when V is a *complex* vector space, then any solvable subalgebra of $\mathfrak{gl}(V)$ is essentially an algebra of upper triangular matrices; i.e., a subalgebra of $T_n(\mathbb{C})$ (wherein we identify an operator T with its matrix $M(T)$ under the isomorphism given above).

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$, and suppose that f is a linear functional on $\mathfrak{g}$. The *joint eigenspace of* $\mathfrak{g}$ *corresponding to* f is the subset of V given by

$$V_f = \{v \in V \mid T(v) = f(T)\, v \text{ for all } T \in \mathfrak{g}\} \tag{4.5}$$

The joint eigenspace V_f is easily shown to be a subspace of V: supposing that $v_1,\, v_2 \in V_f$ and $\alpha \in \mathbb{C}$, then $T(v_1 + v_2) = T(v_1) + T(v_2) = f(T)\, v_1 + f(T)\, v_2 = f(T)\,(v_1 + v_2)$; and similarly, $T(\alpha v_1) = \alpha T(v_1) = \alpha\, f(T)\, v_1 = f(T)\,(\alpha v_1)$, for all $T \in \mathfrak{g}$.

Of course, for a given f, V_f could very well be the trivial subspace $\{0\}$ of V. Any nonzero element of a joint eigenspace of $\mathfrak{g}$ is called a *joint eigenvector* of $\mathfrak{g}$.

Any nonzero vector $v \in V$ which is an eigenvector of each $T \in \mathfrak{g}$ is necessarily a joint eigenvector of $\mathfrak{g}$. For this, we simply define the function $f : \mathfrak{g} \to \mathbb{F}$ by the requirement that

$$T\, v = f(T)\, v,$$

for all $T \in \mathfrak{g}$. It is easy to show that f is a linear functional on $\mathfrak{g}$, and that therefore v is a nonzero element of V_f.

The following important lemma is the key to Lie's Theorem.

Lemma 4.2.1. *(E.B. Dynkin) Let V be a nonzero vector space over $\mathbb{F}$, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, and that f is a linear functional on $\mathfrak{a}$. If V_f is the joint eigenspace of $\mathfrak{a}$ corresponding to f, then V_f is invariant under $\mathfrak{g}$. That is, $X(V_f) \subset V_f$ whenever $X \in \mathfrak{g}$.*

Proof. Let $X \in \mathfrak{g}$ and $v \in V_f$. We want to prove that $X(v) \in V_f$. That is, we want to prove that $T(X(v)) = f(T)\, X(v)$ for any $T \in \mathfrak{a}$. For $v = 0$, this result trivially holds, so we may assume that $v \neq 0$.

Note that for any $T \in \mathfrak{a}$,

$$\begin{aligned} T(X(v)) &= X(T(v)) + (TX - XT)(v) \\ &= X(T(v)) + [T, X](v) \\ &= X(f(T)\, v) + f([T, X])\, v \qquad\qquad \text{(since } [T, X] \in \mathfrak{a}) \\ &= f(T)\, X(v) + f([T, X])\, v \end{aligned} \tag{4.6}$$

The proof will be complete once we prove that $f([T, X]) = 0$.

Let $v_0 = v$, $v_1 = X(v)$, $v_2 = X^2(v), \dots, v_j = X^j(v), \dots$. Next, for each $j \geq 0$, let V_j be the subspace of V spanned by $(v_0, \dots, v_j)$. Since V is finite-dimensional, there is an integer $k \geq 0$ such that $(v_0, \dots, v_k)$ is linearly independent but $(v_0, \dots, v_k, v_{k+1})$ is not. Let k be the smallest such integer.

We claim that for each j such that $0 \leq j \leq k$, the subspace V_j is invariant under *any* $T \in \mathfrak{a}$ and that the matrix of $T|_{V_j}$ with respect to the basis $(v_0, \ldots, v_j)$ of V_j is upper triangular of the form

$$\begin{pmatrix} f(T) & & * \\ & \ddots & \\ 0 & & f(T) \end{pmatrix} \tag{4.7}$$

If $k = 0$, then this is obvious, since $V_j = V_k = V_0 = \mathbb{F}v_0$, and $T(v_0) = f(T)\, v_0$, because $v_0 \in V_f$.

So assume that $k \geq 1$. Equation (4.6) says that for any $T \in \mathfrak{a}$,

$$T(v_1) = f(T)\, v_1 + f([T, X])\, v_0,$$

which shows that the subspace $V_1 = \mathbb{F}v_0 + \mathbb{F}v_1$ is invariant under T. Moreover, relative to the basis $(v_0,\, v_1)$ of V_1, the matrix of the restriction $T|_{V_1}$ is

$$\begin{pmatrix} f(T) & f([T, X]) \\ 0 & f(T) \end{pmatrix}$$

We will now use induction on j to prove the same thing for V_j, for any $j \leq k$. So assume that V_{j-1} is T-invariant, and that, for any $T \in \mathfrak{a}$, the matrix of the restriction $T|_{V_{j-1}}$ with respect to the basis $(v_0, \ldots, v_{j-1})$ of V_{j-1} is of the form 4.7. Now for any $T \in \mathfrak{a}$, we have,

$$\begin{aligned} T(v_j) &= T(X^j(v)) \\ &= T\, X\, (X^{j-1}(v)) \\ &= X\, T\, (X^{j-1}(v)) + [T, X]\, X^{j-1}(v) \\ &= X\, T(v_{j-1}) + [T, X](v_{j-1}) \\ &= X\big(f(T)v_{j-1} + \sum_{i<j-1} c_i v_i\big) \;+\; \big(f([T, X])\, v_{j-1} + \sum_{i<j-1} d_i v_i\big), \end{aligned}$$

by the induction hypothesis, where the c_i and the d_i are constants. The last expression above then equals

$$\begin{aligned} f(T)\, X(v_{j-1}) + \sum_{i<j-1} c_i X(v_i) + f([T, X])\, v_{j-1} + \sum_{i<j-1} d_i v_i & \\ = f(T)\, v_j + \sum_{i<j-1} c_i v_{i+1} + f([T, X])\, v_{j-1} + \sum_{i<j-1} d_i v_i & \\ = f(T)\, v_j + \big(\text{a linear combination of } (v_0, \ldots, v_{j-1})\big) & \end{aligned}$$

This proves our claim. In particular, V_k is invariant under any $T \in \mathfrak{a}$, and the matrix of $T|_{V_k}$ is of the form 4.7.

This means that for any $T \in \mathfrak{a}$, the trace of $T|_{V_k}$ is $(k+1)f(T)$. Hence, the trace of the restriction $[T,X]|_{V_k}$ is $(k+1)\,f([T,X])$. But then, this trace also equals

$$\operatorname{tr}(TX - XT)|_{V_k} = \operatorname{tr}(T|_{V_k}\,X|_{V_k}) - \operatorname{tr}(X|_{V_k}\,T|_{V_k}) = 0.$$

Thus $(k+1)\,f([T,X]) = 0$, whence $f([T,X]) = 0$, proving the lemma. □

The following theorem can be construed as a generalization of Theorem 1.5.2, which states that any linear operator on a complex vector space has an eigenvector.

Theorem 4.2.2. *Let V be a nonzero vector space over $\mathbb{C}$, and let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ has a joint eigenvector.*

This theorem asserts that there exists a nonzero vector $v \in V$ and a linear functional f on $\mathfrak{g}$ such that $T(v) = f(T)\,v$, for all $T \in \mathfrak{g}$.

Proof. We prove the theorem by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \mathbb{C}T$, where T is a linear operator on V. By Theorem 1.5.2, T has an eigenvalue λ. Let v be an eigenvector corresponding to λ. For any $S \in \mathfrak{g}$, we have $S = cT$, so $S(v) = cT(v) = c\lambda\, v$, so we can put $f(cT) = c\lambda$. Clearly, $f \in \mathfrak{g}^*$.

Now assume that $\dim \mathfrak{g} = m > 1$, and that any solvable Lie subalgebra of $\mathfrak{gl}(V)$ of dimension $< m$ has a joint eigenvector. Consider the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}'$ is a proper ideal of $\mathfrak{g}$, so it is a subalgebra of $\mathfrak{gl}(V)$ of dimension $< m$.

Next let $\mathfrak{h}$ be *any* vector subspace of $\mathfrak{g}$, of dimension $m-1$, such that $\mathfrak{g}' \subset \mathfrak{h}$. Such an $\mathfrak{h}$ of course exists. By Proposition 4.1.4, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Moreover, since $\mathfrak{g}$ is solvable, so is $\mathfrak{h}$. (See the observations made after Example 4.1.9.)

Thus, by the induction hypothesis, $\mathfrak{h}$ has a joint eigenvector. In other words, $\mathfrak{h}$ has a nonzero joint eigenspace V_μ, where μ is a linear functional on $\mathfrak{h}$.

Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, we conclude, using Lemma 4.2.1, that V_μ must be $\mathfrak{g}$-invariant. Let S be a nonzero element of $\mathfrak{g}$ not in $\mathfrak{h}$. Then, since $\dim \mathfrak{h} = m-1$, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}S$. The subspace V_μ is S-invariant, so the restriction $S|_{V_\mu}$ must have an eigenvalue λ. Let $v \in V_\mu$ be an eigenvector of $S|_{V_\mu}$ corresponding to λ.

For any $T \in \mathfrak{g}$, we have $T = cS + Y$, for unique $Y \in \mathfrak{h}$ and $c \in \mathbb{C}$. Define the map $f : \mathfrak{g} \to \mathbb{C}$ by $f(cS + Y) = c\lambda + \mu(Y)$. It is easy to prove that f is a linear functional on $\mathfrak{g}$. Moreover, if $T = cS + Y \in \mathfrak{g}$,

$$\begin{aligned} T(v) &= (cS + Y)(v) \\ &= c\,S(v) + Y(v) \\ &= c\lambda\, v + \mu(Y)\, v \\ &= (c\lambda + \mu(Y))\, v \\ &= f(T)\, v. \end{aligned}$$

This shows that v is a joint eigenvector of $\mathfrak{g}$, completing the induction step and proving the theorem. □

Theorem 4.2.3. *(Lie's Theorem) Let V be a nonzero complex vector space, and let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then V has a basis $(v_1, \ldots, v_n)$ with respect to which every element of $\mathfrak{g}$ has an upper triangular matrix.*

Proof. The proof is by induction on $\dim V$. If $\dim V = 1$, there is nothing to prove. So assume that $\dim V = n > 1$, and that Lie's theorem holds for all complex vector spaces of dimension $< n$.

Now by Theorem 4.2.2, $\mathfrak{g}$ has a joint eigenvector v_1. Let $V_1 = \mathbb{C}\, v_1$. Then, for every $T \in \mathfrak{g}$, the subspace V_1 is T-invariant; let $\widetilde{T} : V/V_1 \to V/V_1$ be the induced linear map.

The map $T \mapsto \widetilde{T}$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(V/V_1)$. It's clearly linear, and the relation $\widetilde{[S,T]} = [\widetilde{S}, \widetilde{T}]$ is easily verified by a simple computation. Since homomorphic images of solvable Lie algebras are solvable, the image $\widetilde{\mathfrak{g}}$ of this homomorphism is a solvable Lie subalgebra of $\mathfrak{gl}(V/V_1)$.

Since $\dim(V/V_1) = n - 1$, we can now apply the induction hypothesis to obtain a basis $(v_2 + V_1, \ldots, v_n + V_1)$ of V/V_1 for which the elements of $\widetilde{\mathfrak{g}}$ are upper triangular.

The list $(v_1, v_2, \ldots, v_n)$ is then a basis of V. For each $T \in \mathfrak{g}$, the matrix of $\widetilde{T} : V/V_1 \to V/V_1$ with respect to $(v_2 + V_1, \ldots, v_n + V_1)$ is upper triangular. Hence the matrix of T with respect to $(v_1, v_2, \ldots, v_n)$ is upper triangular, proving the theorem. □

A *flag* in a vector space V is a sequence $(V_1, \ldots, V_k)$ of subspaces of V such that $V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_k$. We say that a linear operator $T \in \mathcal{L}(V)$ *stabilizes* the flag $(V_1, \ldots, V_k)$ if each V_i is T-invariant. Finally, a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ *stabilizes* the flag $(V_1, \ldots, V_k)$ if each $T \in \mathfrak{g}$ stabilizes the flag.

Corollary 4.2.4. *If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, then $\mathfrak{g}$ stabilizes some flag $(\{0\} = V_0, V_1, V_2, \ldots, V_n = V)$.*

Proof. Let $(v_1, \ldots, v_n)$ be a basis of V with respect to which the matrix of every element of $\mathfrak{g}$ is upper triangular. Then, for each i, let $V_i = \mathbb{C}v_1 + \cdots + \mathbb{C}v_i$. $\square$

Corollary 4.2.5. *(Lie's Abstract Theorem) Let $\mathfrak{g}$ be a solvable Lie algebra over $\mathbb{C}$, of dimension N. Then $\mathfrak{g}$ has a chain of ideals $\{0\} = \mathfrak{g}_0 \subsetneqq \mathfrak{g}_1 \subsetneqq \cdots \subsetneqq \mathfrak{g}_N = \mathfrak{g}$.*

Proof. The adjoint representation $x \mapsto \operatorname{ad} x$ maps $\mathfrak{g}$ onto the solvable Lie subalgebra $\operatorname{ad} \mathfrak{g}$ of $\operatorname{Der} \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$. Thus $\operatorname{ad} \mathfrak{g}$ stabilizes a flag $\{0\} = \mathfrak{g}_0 \subsetneqq \mathfrak{g}_1 \subsetneqq \cdots \subsetneqq \mathfrak{g}_N = \mathfrak{g}$ in $\mathfrak{g}$. Each subspace $\mathfrak{g}_i$ therefore satisfies $\operatorname{ad} x\,(\mathfrak{g}_i) \subset \mathfrak{g}_i$, for all $x \in \mathfrak{g}$. This means that $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. $\square$

In particular, Corollary 4.2.5 shows that if $\mathfrak{g}$ is a complex solvable Lie algebra and if $0 \leq i \leq \dim \mathfrak{g}$, then $\mathfrak{g}$ has an ideal of dimension i.

In Example 4.1.9, we saw that the Lie algebra $T_n(\mathbb{F})$ of all upper triangular $n \times n$ matrices over $\mathbb{F}$ is solvable. If a Lie algebra $\mathfrak{g}$ is solvable and complex, then the following shows that $\mathfrak{g}$ is in some sense just a subalgebra of $T_n(\mathbb{C})$. Thus $T_n(\mathbb{C})$ is *the* "prototypical" solvable complex Lie algebra. For this, we will need the following important theorem.

Theorem 4.2.6. *(Ado's Theorem) Let $\mathfrak{g}$ be any nonzero Lie algebra over $\mathbb{F}$. Then there exists a vector space V over $\mathbb{F}$ and an injective Lie algebra homomorhism φ of $\mathfrak{g}$ into $\mathfrak{gl}(V)$.*

We won't be needing Ado's Theorem in the rest of this book, so we omit its proof.

Now suppose that $\mathfrak{g}$ is a solvable complex Lie algebra. Using Ado's Theorem, we may therefore identify $\mathfrak{g}$ with a (solvable) Lie subalgebra of $\mathfrak{gl}(V)$. Then, from Lie's theorem, there is a basis B of V with respect to which the matrix of every element of $\mathfrak{g}$ is upper triangular. Now, for every linear operator T on V, let $M(T)$ be its matrix with respect to B. Then the map $T \mapsto M(T)$ is a Lie algebra isomorphism of $\mathfrak{gl}(V)$ onto $\mathfrak{gl}(n, \mathbb{C})$. The image of $\mathfrak{g}$ under this isomorphism is a Lie subalgebra of $T_n(\mathbb{C})$. Thus $\mathfrak{g}$ may be identified with this Lie subalgebra of $T_n(\mathbb{C})$.

Bibliography

[Axl97] S. Axler, *Linear Algebra Done Right*, second ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997. MR 1482226

[Dau85] J. W. Dauben, *The History of Mathematics from Antiquity to the Present*, Bibliographies of the History of Science and Technology, vol. 6, Garland Publishing, Inc., New York, 1985, A selective bibliography, Garland Reference Library of the Humanities, 313. MR 790680

[Jac79] N. Jacobson, *Lie algebras*, Dover Publications, Inc., New York, 1979, Republication of the 1962 original. MR 559927

[Rud91] W. Rudin, *Functional Analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR 1157815

[Sch51] E. Schenkman, *A theory of subinvariant Lie algebras*, Amer. J. Math. **73** (1951), 453–474. MR 42399