Chapter 3

Basic Algebraic Facts

In this section we explore the basic algebraic properties satisfied by all Lie algebras. Many of these properties, properly formulated, are shared by general algebras.

3.1 Structure Constants.

Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{F}$, and that $B = (x_1, \dots, x_n)$ is a basis of $\mathfrak{g}$. Then there exist unique scalars c_{ij}^k $(1 \le k \le n)$ such that

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k \tag{3.1}$$

for all $1 \le i, j \le n$. The scalars c_{ij}^k are called the *structure constants of* $\mathfrak{g}$ *relative to the given basis* B. Since every element of $\mathfrak{g}$ is a unique linear combination of the basis vectors in B, we see that the structure constants completely determine the Lie bracket $[x, y]$, for any x, $y \in \mathfrak{g}$.

From the anticommutativity (2.3) and the Jacobi identity (2.4), we see that the structure constants c_{ij}^k satsify the relations

$$c_{ji}^k = -c_{ij}^k \tag{3.2}$$

$$\sum_{r=1}^{n} (c_{ir}^m c_{jk}^r + c_{jr}^m c_{ki}^r + c_{kr}^m c_{ij}^r) = 0, \tag{3.3}$$

for all i, j, k, m.

Conversely, suppose that there exist n^3 constants c_{ij}^k in $\mathbb{F}$ satisfying the relations (3.2) and (3.3). Then it can be shown, by a straightforward computation, that if $\mathfrak{g}$ is a vector space with basis $B = (x_1, \dots, x_n)$ and we define a bilinear binary operation $[\ ,\ ]$ on $\mathfrak{g}$ via (3.1), then this binary operation is anticommutative and satisfies the Jacobi identity.

Thus a Lie algebra $\mathfrak{g}$ is completely determined by its structure constants and the relations (3.1) on a given basis $(x_1, \dots, x_n)$.

In this course, we will not be making much use of structure constants.

3.2 Quotient Algebras, Homomorphisms, and Isomorphisms.

Let $\mathfrak{h}$ and $\mathfrak{u}$ be subalgebras of a Lie algebra $\mathfrak{g}$. Then the subspace $\mathfrak{h} \cap \mathfrak{u}$ is easily checked to be a Lie subalgebra of $\mathfrak{g}$. In addition, if one of them is an ideal of $\mathfrak{g}$, then $\mathfrak{h} + \mathfrak{u}$ is a subalgebra of $\mathfrak{g}$.

If U and W are nonempty subsets of $\mathfrak{g}$, we define $[U, W]$ to be the subspace spanned by all $[u, w]$, where $u \in U$, $w \in W$. Thus a subspace U of $\mathfrak{g}$ is a subalgebra if and only if $[U, U] \subset U$.

Let $\mathfrak{a}$ be an ideal of a Lie algebra $\mathfrak{g}$. The quotient space $\mathfrak{g}/\mathfrak{a}$ has a (natural) Lie algebra structure, in which the Lie bracket is defined by

$$[x + \mathfrak{a}, y + \mathfrak{a}] = [x, y] + \mathfrak{a}. \tag{3.4}$$

The binary operation on $\mathfrak{g}/\mathfrak{a}$ given by 3.4 is well-defined: if $x + \mathfrak{a} = x_1 + \mathfrak{a}$ and $y + \mathfrak{a} = y_1 + \mathfrak{a}$, then $x_1 - x \in \mathfrak{a}$ and $y_1 - y \in \mathfrak{a}$, so

$$\begin{aligned}[x_1, y_1] &= [x + (x_1 - x), y + (y_1 - y)] \\ &= [x, y] + [x_1 - x, y] + x, y_1 - y] + [x_1 - x, y_1 - y].\end{aligned}$$

Since the last three terms above belong to $\mathfrak{a}$, we therefore have $[x_1, y_1] + \mathfrak{a} = [x, y] + \mathfrak{a}$. It is easy to verify that the binary operation (3.4) on $\mathfrak{g}/\mathfrak{a}$ is a Lie bracket on $\mathfrak{g}/\mathfrak{a}$. We call the quotient space $\mathfrak{g}/\mathfrak{a}$, equipped with this Lie bracket, the *quotient Lie algebra of* $\mathfrak{g}$ *by* $\mathfrak{a}$.

Example 3.2.1. We saw in Example 2.1.14 that $\mathfrak{sl}(n, \mathbb{F})$ is an ideal of $\mathfrak{gl}(n, \mathbb{F})$, of dimension $n^2 - 1$. The quotient Lie algebra $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{F})/\mathfrak{sl}(n, \mathbb{F})$ is one-dimensional, and so must be abelian. The fact that $\mathfrak{k}$ is abelian is also easy to see because $[X, Y] \in \mathfrak{sl}(n, \mathbb{F})$ for all X, $Y \in \mathfrak{gl}(n, \mathbb{F})$.

Let $\mathfrak{g}$ and $\mathfrak{m}$ be Lie algebras. A *homomorphism* from $\mathfrak{g}$ to $\mathfrak{m}$ is a linear map $\varphi : \mathfrak{g} \to \mathfrak{m}$ such that

$$\varphi[x, y] = [\varphi(x), \varphi(y)],$$

for all x, $y \in g$. An *isomorphism* is a one-to-one, onto, homomorphism. We say that Lie algebras $\mathfrak{g}$ and $\mathfrak{m}$ are *isomorphic*, written $\mathfrak{g} \cong \mathfrak{m}$, if there exists an isomorphism $\varphi : \mathfrak{g} \to \mathfrak{m}$. An *automorphism* of $\mathfrak{g}$ is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}$.

As an example, if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the natural projection

$$\begin{aligned} \pi : & \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} \\ & x \mapsto x + \mathfrak{a} \end{aligned}$$

is a surjective Lie algebra homomorphism.

If $\varphi : \mathfrak{g} \to \mathfrak{m}$ is a homomorphism, then $\ker \varphi$ is an ideal of $\mathfrak{g}$. In fact, if $x \in \ker \varphi$ and $y \in \mathfrak{g}$, then

$$\varphi[x, y] = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0,$$

so $[x, y] \in \ker \varphi$.

Exercise 3.2.2.

Exercise 3.2.3.

Theorem 3.2.4. *(The Homomorphism Theorem) Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the natural projection. Suppose that $\varphi : \mathfrak{g} \to \mathfrak{m}$ is a homomorphism such that $\mathfrak{a} \subset \ker \varphi$. Then there is a unique homomorphism $\widetilde{\varphi} : \mathfrak{g}/\mathfrak{a} \to \mathfrak{m}$ satisfying $\widetilde{\varphi} \circ \pi = \varphi$.*

Proof. Let $\widetilde{\varphi} : \mathfrak{g}/\mathfrak{a} \to \mathfrak{m}$ be given by $\widetilde{\varphi}(x + \mathfrak{a}) = \varphi(x)$, for all $x \in \mathfrak{a}$. Since $\varphi(\mathfrak{a}) = 0$, $\widetilde{\varphi}$ is well-defined. It is a homomorphism since $\widetilde{\varphi}[x + \mathfrak{a}, y + \mathfrak{a}] = \widetilde{\varphi}([x, y]+\mathfrak{a}) = \varphi[x, y] = [\varphi(x), \varphi(y)] = [\widetilde{\varphi}(x+\mathfrak{a}), \widetilde{\varphi}(y+\mathfrak{a})]$. And it is unique, since the condition $\widetilde{\varphi}\circ\pi = \varphi$ means that, for all $x \in \mathfrak{g}$, $\widetilde{\varphi}(x+\mathfrak{a}) = \widetilde{\varphi}\circ\pi(x) = \varphi(x)$. □

Corollary 3.2.5. *Let $\varphi : \mathfrak{g} \to \mathfrak{m}$ be a Lie algebra homomorphism. Then the image $\varphi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{m}$, and the resulting map $\widetilde{\varphi} : \mathfrak{g}/\ker \varphi \to \varphi(\mathfrak{g})$ is a Lie algebra isomorphism. Thus, if φ is onto, then $\widetilde{\varphi}$ is an isomorphism of $\mathfrak{g}/\ker \varphi$ onto $\mathfrak{m}$.*

Proof. For any x, $y \in \mathfrak{g}$, we have $[\varphi(x), \varphi(y)] = \varphi[x, y] \in \varphi(\mathfrak{g})$, so $\varphi(\mathfrak{g})$ is a subalgebra of $\mathfrak{m}$. Put $\mathfrak{a} = \ker \varphi$ in Theorem 3.2.4. Then $\widetilde{\varphi}$ is injective since if $\widetilde{\varphi}(x + \mathfrak{a}) = 0$, then $\varphi(x) = 0$, so $x \in \mathfrak{a}$, and thus $x + \mathfrak{a} = \mathfrak{a}$. Thus the map $\widetilde{\varphi} : \mathfrak{g}/\ker \varphi \to \varphi(\mathfrak{g})$ is an isomorphism. □

If $\ker \varphi = \{0\}$, then $\mathfrak{g}$ is isomorphic to its image $\varphi(\mathfrak{g})$ in $\mathfrak{m}$. In this case, we say that $\mathfrak{g}$ is *embedded in* $\mathfrak{m}$.

Theorem 3.2.6. *(The Correspondence Theorem) Let $\varphi : \mathfrak{g} \to \mathfrak{m}$ be a surjective homomorphism.*

1. *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\varphi(\mathfrak{a})$ is an ideal of $\mathfrak{m}$.*

2. *If $\mathfrak{s}$ is an ideal of $\mathfrak{m}$, the $\varphi^{-1}(\mathfrak{s})$ is an ideal of $\mathfrak{g}$ which contains $\ker\varphi$.*

3. *The mappings $\mathfrak{a} \mapsto \varphi(\mathfrak{a})$ and $\mathfrak{s} \mapsto \varphi^{-1}(\mathfrak{s})$ are inverse mappings between the set of all ideals of $\mathfrak{g}$ which contain $\ker\varphi$ and the set of all ideals of $\mathfrak{m}$, so the two sets of ideals are in one-to-one correspondence.*

4. *$\mathfrak{g}/\mathfrak{a} \cong \mathfrak{m}/\varphi(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of $\mathfrak{g}$ containing $\ker\varphi$.*

5. *The correspondence in (3) preserves inclusion:*
$$\ker\varphi \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \iff \varphi(\mathfrak{a}_1) \subset \varphi(\mathfrak{a}_2).$$

Proof.

1. For any $y \in \mathfrak{m}$ and $v \in \mathfrak{a}$, we have $y = \varphi(x)$ for some $x \in \mathfrak{g}$, so
$$\begin{aligned}[y, \varphi(v)] &= [\varphi(x), \varphi(v)] \\ &= \varphi[x, v] \in \varphi(\mathfrak{a}).\end{aligned}$$
Hence $[\mathfrak{m}, \varphi(\mathfrak{a})] \subset \varphi(\mathfrak{a})$, and $\varphi(\mathfrak{a})$ is an ideal of $\mathfrak{m}$.

2. Let $v \in \varphi^{-1}(\mathfrak{s})$. Then for any $x \in \mathfrak{g}$, we have
$$\varphi[x, v] = [\varphi(x), \varphi(v)] \in [\mathfrak{m}, \mathfrak{s}] \subset \mathfrak{s},$$
so $[x, v] \in \varphi^{-1}(\mathfrak{s})$.

3. We first claim that if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ containing $\ker\varphi$, then $\varphi^{-1}(\varphi(\mathfrak{a})) = \mathfrak{a}$. Since clearly $\mathfrak{a} \subset \varphi^{-1}(\varphi(\mathfrak{a}))$, it suffices to prove that $\varphi^{-1}(\varphi(\mathfrak{a})) \subset \mathfrak{a}$. But if $x \in \varphi^{-1}(\varphi(\mathfrak{a}))$, then $\varphi(x) = \varphi(v)$ for some $v \in \mathfrak{a}$, so $x - v \in \ker\varphi$, and hence $x \in v + \ker\varphi \subset \mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.

   Next, it is clear from the surjectivity of φ that if A is any subset of $\mathfrak{m}$, then $\varphi(\varphi^{-1}(A)) = A$. Thus, in particular, if $\mathfrak{s}$ is an ideal of $\mathfrak{m}$, then $\varphi(\varphi^{-1}(\mathfrak{s})) = \mathfrak{s}$.

   From the above, we see that $\mathfrak{a} \mapsto \varphi(\mathfrak{a})$ is a bijection between the sets in question, with inverse $\mathfrak{s} \mapsto \varphi^{-1}(\mathfrak{s})$.

4. Consider the following diagram of homomorphisms
$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \varphi\ } & \mathfrak{m} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{g}/\mathfrak{a} & \xrightarrow[\ \varphi_1\ ]{} & \mathfrak{m}/\varphi(\mathfrak{a}) \end{array} \tag{3.5}$$
where π and p are projections. Now $p \circ \varphi$ is a homomorphism of $\mathfrak{g}$ onto $\mathfrak{m}/\varphi(\mathfrak{a})$ whose kernel is $\varphi^{-1}(\varphi(\mathfrak{a})) = \mathfrak{a}$. Hence by the Homomorphism Theorem (Theorem 3.2.4)and its corollary, there is a isomorphism φ_1 from $\mathfrak{g}/\mathfrak{a}$ onto $\mathfrak{m}/\varphi(\mathfrak{a})$ such that $\varphi_1 \circ \pi = p \circ \varphi$.

5. Obvious.

□

Theorem 3.2.7. *(The Isomorphism Theorem) If $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{s}$ a subalgebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$, then*

1. *$\mathfrak{s} \cap \mathfrak{a}$ is an ideal of $\mathfrak{s}$*
2. *$\mathfrak{s}/(\mathfrak{s} \cap \mathfrak{a}) \cong (\mathfrak{s} + \mathfrak{a})/\mathfrak{a}$*
3. *If $\mathfrak{b}$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{b} \subset \mathfrak{a}$, then $(\mathfrak{g}/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b}) \cong \mathfrak{g}/\mathfrak{a}$.*

Proof. 1. Easy.

2. We already know that $\mathfrak{s} + \mathfrak{a}$ is a subalgebra of $\mathfrak{g}$. Consider the diagram

$$\begin{array}{ccc} \mathfrak{s} & \xrightarrow{\ \pi\ } & \mathfrak{s}/(\mathfrak{s} \cap \mathfrak{a}) \\ {\scriptstyle i}\downarrow & & \vdots\downarrow{\scriptstyle i'} \\ \mathfrak{s} + \mathfrak{a} & \xrightarrow{\ p\ } & (\mathfrak{s} + \mathfrak{a})/\mathfrak{a} \end{array}$$

where π and p are projections, and i is the inclusion map of $\mathfrak{s}$ into $\mathfrak{s} + \mathfrak{a}$. $p \circ i$ is obviously a homomorphism, and it is surjective, since any element of $(\mathfrak{s} + \mathfrak{a})/\mathfrak{a}$ is of the form $v + w + \mathfrak{a}$, where $v \in \mathfrak{s}$ and $w \in \mathfrak{a}$, and this element is of course the same as $v + \mathfrak{a} = p \circ i(v)$. The kernel of $p \circ i$ is $\{v \in \mathfrak{s} \,|\, v + \mathfrak{a} = \mathfrak{a}\} = \{v \in \mathfrak{s} \,|\, v \in \mathfrak{a}\} = \mathfrak{s} \cap \mathfrak{a}$. Thus by the Homomorphism Theorem, the resulting map $i' : \mathfrak{s}/(\mathfrak{s} \cap \mathfrak{a}) \to (\mathfrak{s} + \mathfrak{a})/\mathfrak{a}$ is an isomorphism.

3. Consider the map $h : \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{a}$ given by $x + \mathfrak{b} \mapsto x + \mathfrak{a}$. h is well-defined, since $\mathfrak{b} \subset \mathfrak{a}$ and is easily checked to be a surjective Lie algebra homomorphism. Its kernel is the ideal $\{x + \mathfrak{b} \,|\, x + \mathfrak{a} = \mathfrak{a}\} = \{x + \mathfrak{b} \,|\, x \in \mathfrak{a}\} = \mathfrak{a}/\mathfrak{b}$ of $\mathfrak{g}/\mathfrak{b}$. Thus, by the Homomorphism Theorem, the algebras $(\mathfrak{g}/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b})$ and $\mathfrak{g}/\mathfrak{a}$ are isomorphic.

□

Exercise 3.2.8. (Graduate Exercise.) Suppose that φ is an *involution* of a Lie algebra $\mathfrak{g}$; i.e., an automorphism φ of $\mathfrak{g}$ such that $\varphi^2 = I_{\mathfrak{g}}$. Let $\mathfrak{h} = \{x \in \mathfrak{g} \,|\, \varphi(x) = x\}$ be the $+1$ eigenspace and $\mathfrak{q} = \{x \in \mathfrak{g} \,|\, \varphi(x) = -x\}$ the -1-eigenspace of φ, respectively. Prove that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, and that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ (so that $\mathfrak{h}$ is a subalgebra), $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

Definition 3.2.9. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. V is called the *representation space* of π. If there is a representation of $\mathfrak{g}$ on V, then we say that $\mathfrak{g}$ *acts on* V.

If π is a representation of $\mathfrak{g}$ on V, then of course $\pi(x)$ is a linear map on V for any $x \in \mathfrak{g}$.

Example 3.2.10. If V is any vector space, then $\mathfrak{gl}(V)$ acts on V, via the identity map $\mathrm{id} : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$. Any Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ likewise acts on V, via the inclusion map $\iota : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$. This action is called the *standard representation* of $\mathfrak{g}$ on V.

Note that $\mathfrak{sl}(n, \mathbb{F})$, $\mathfrak{so}(n, \mathbb{F})$ and $\mathfrak{sp}(n, \mathbb{F})$ ((Examples 2.1.14), 2.1.17, and 2.1.18, respectively) are Lie algebras of matrices acting on $\mathbb{F}^n$, $\mathbb{F}^n$, and $\mathbb{F}^{2n}$ under their respective standard representations.

Example 3.2.11. The *trivial representation* of $\mathfrak{g}$ on V is the map $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\pi(x) = 0$, for all $x \in \mathfrak{g}$.

In *representation theory*, one studies representations of Lie algebras, and their associated Lie groups, on finite and infinite-dimensional vector spaces. Representation theory has intimate connections to number theory, physics, differential and symplectic geometry, and harmonic and geometric analysis, to mention but a few fields. It is an extremely active and vibrant field of mathematics.

3.3 Centers, Centralizers, Normalizers, and Simple Lie Algebras

The *center* of a Lie algebra $\mathfrak{g}$ is the set $\mathfrak{c} = \{c \in \mathfrak{g} \,|\, [c, v] = 0 \text{ for all } v \in \mathfrak{g}\}$. It is obvious that $\mathfrak{c}$ is an ideal of $\mathfrak{g}$. If A is a nonempty subset of $\mathfrak{g}$, then the *centralizer* of A is the set $\mathfrak{c}(A) = \{x \in \mathfrak{g} \,|\, [x, v] = 0 \text{ for all } v \in A\}$. It is easily checked that $\mathfrak{c}(A)$ is a subspace of $\mathfrak{g}$. Note that $\mathfrak{c} = \mathfrak{c}(\mathfrak{g})$.

Proposition 3.3.1. *Let A be a nonempty subset of $\mathfrak{g}$. Then its centralizer $\mathfrak{c}(A)$ is a subalgebra of $\mathfrak{g}$.*

Proof. This is an immediate consequence of the Jacobi identity. Let x, $y \in \mathfrak{c}(A)$ and let $a \in A$. Then

$$[[x, y], a] = -[[y, a], x] - [[a, x], y] = -[0, x] - [0, y] = 0.$$

□

Proposition 3.3.2. *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then its centralizer $\mathfrak{c}(\mathfrak{a})$ is an ideal of $\mathfrak{g}$.*

Proof. Another immediate consequence of the Jacobi identity: let $c \in \mathfrak{c}(\mathfrak{a})$, $v \in \mathfrak{g}$, and $x \in \mathfrak{a}$. Then

$$[[c, v], x] = -[[v, x], c] - [[x, c], v].$$

But $c \in \mathfrak{c}$, so $[x, c] = 0$ and $[v, x] \in \mathfrak{a}$ so $[[v, x], c] = 0$. □

Proposition 3.3.3. *Let φ be a surjective Lie algebra homomorphism of $\mathfrak{g}$ onto $\mathfrak{m}$. If $\mathfrak{c}$ denotes the center of $\mathfrak{g}$, then its image $\varphi(\mathfrak{c})$ lies in the center of $\mathfrak{m}$.*

Proof. $\mathfrak{m} = \varphi(\mathfrak{g})$, so $[\mathfrak{m}, \varphi(\mathfrak{c})] = [\varphi(\mathfrak{g}), \varphi(\mathfrak{c})] = \varphi([\mathfrak{g}, \mathfrak{c}]) = \varphi(\{0\}) = \{0\}$. □

If $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$, its *normalizer* is the set $\mathfrak{n}(\mathfrak{s}) = \{x \in \mathfrak{g} \,|\, [x, v] \in \mathfrak{s} \text{ for all } v \in \mathfrak{s}\}$. $\mathfrak{n}(\mathfrak{s})$ is clearly a subspace of $\mathfrak{g}$ containing $\mathfrak{s}$, and the Jacobi identity shows that it is in fact a subalgebra of $\mathfrak{g}$:

Proposition 3.3.4. $\mathfrak{n}(\mathfrak{s})$ *is a subalgebra of* $\mathfrak{g}$.

Proof. Let x and y be in $\mathfrak{n}(\mathfrak{s})$, and let $s \in \mathfrak{s}$. Then

$$[[x, y], s] = -[[y, s], x] - [[s, x], y] \;\in\; -[\mathfrak{s}, x] - [\mathfrak{s}, y] \;\subset \mathfrak{s} + \mathfrak{s} = \mathfrak{s}.$$

□

One more thing: it is easy to see that $\mathfrak{n}(\mathfrak{s})$ is the largest subalgebra of $\mathfrak{g}$ for which $\mathfrak{s}$ is an ideal.

Example 3.3.5. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \{X \in \mathfrak{gl}(2, \mathbb{C}) \,|\, \mathrm{tr}(X) = 0\}$. Its *standard basis* is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

The commutation relations among these basis elements is given by

$$\begin{aligned} [h, e] &= he - eh = 2e \\ [h, f] &= hf - fh = -2f \\ [e, f] &= ef - fe = h. \end{aligned} \tag{3.7}$$

From the above, we see that the one-dimensional algebra $\mathbb{C}h$ is its own normalizer. ($\mathbb{C}h$ is called a *Cartan subalgebra* of $\mathfrak{g}$.) We will see later on that the commutation relations (3.7) play a key role in the structure and representation theory of semisimple Lie algebras.

Definition 3.3.6. A Lie algebra $\mathfrak{g}$ is said to be *simple* if $\mathfrak{g}$ is non-abelian and $\mathfrak{g}$ has no ideals except $\{0\}$ and $\mathfrak{g}$.

Example 3.3.7. $\mathfrak{sl}(2, \mathbb{C})$ is simple. Suppose that $\mathfrak{a} \neq \{0\}$ is an ideal of $\mathfrak{sl}(2, \mathbb{C})$. Let $v \neq 0$ be an element of $\mathfrak{a}$, and write $v = \alpha e + \beta f + \gamma h$, where not all of α, β or γ are 0.

Assume that $\alpha \neq 0$. Then from the commutation relations (3.7), $[v, f] = \alpha h - 2\gamma f \in \mathfrak{a}$, and so $[[v, f], f] = -2\alpha f \in \mathfrak{a}$. Hence $f \in \mathfrak{a}$, and so $h = [e, f] \in \mathfrak{a}$ and also $e = 1/2[h, e] \in \mathfrak{a}$. Thus $\alpha \neq 0$ implies $\mathfrak{a} = \mathfrak{g}$. A similar argument shows that $\beta \neq 0$ implies $\mathfrak{a} = \mathfrak{g}$.

Finally, if $\gamma \neq 0$, then $[v, e] = -\beta h + 2\gamma e \in \mathfrak{a}$, so the argument in the preceding paragraph shows that $\mathfrak{a} = \mathfrak{g}$.

One of the goals of this course is to obtain Cartan's classification of all simple Lie algebras over $\mathbb{C}$. These consist of four (infinite) classes – the so-called classical simple Lie algebras – and five so-called exceptional Lie algebras.

3.4 The Adjoint Representation.

For each $x \in \mathfrak{g}$ define the linear operator $\text{ad } x : \mathfrak{g} \to \mathfrak{g}$ by $\text{ad } x\ (y) = [x, y]$, for all $y \in \mathfrak{g}$. The Jacobi identity shows that $\text{ad } x$ is a derivation of $\mathfrak{g}$, since for all $u,\ v \in \mathfrak{g}$, we have

$$\begin{aligned}\text{ad } x\ [u, v] &= [x, [u, v]] \\ &= -[u, [v, x]] - [v, [x, u]] \\ &= [u, [x, v]] + [[x, u], v] \\ &= [u, \text{ad } x\ (v)] + [\text{ad } x\ (u), v].\end{aligned}$$

Example 3.4.1. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with standard basis (e, f, h) given by (3.6). Using the commutation relations (3.7), we see that the matrices of $\text{ad}\, e$, $\text{ad}\, f$, and $\text{ad}\, h$ with respect to the standard basis are:

$$\text{ad}\, e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \text{ad}\, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \ \text{ad}\, h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proposition 3.4.2. *The map $x \mapsto \text{ad}\, x$ is a homomorphism of $\mathfrak{g}$ into the Lie algebra* Der $\mathfrak{g}$.

Proof. First, we show that $\text{ad } (x + y) = \text{ad } x + \text{ad } y$. Now for all $z \in \mathfrak{g}$, we have

$$\begin{aligned}\text{ad } (x + y)\ (z) &= [x + y, z] = [x, z] + [y, z] \\ &= \text{ad } x\ (z) + \text{ad } y\ (z) \\ &= (\text{ad } x + \text{ad } y)(z).\end{aligned}$$

Similarly, $\text{ad } (\alpha x) = \alpha(\text{ad } x)$, since for all $z \in \mathfrak{g}$, we have $\text{ad } (\alpha x)\ (z) = [\alpha x, z] = \alpha[x, z] = \alpha \text{ ad } x\ (z)$.

Finally, we prove that ad $[x, y] = [\text{ad } x, \text{ad } y]$ for all x and y in $\mathfrak{g}$. For any $z \in \mathfrak{g}$, we have

$$\begin{aligned} \text{ad } [x, y]\ (z) &= [[x, y], z] \\ &= -[[y, z], x] - [[z, x], y] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= \text{ad } x\ (\text{ad } y\ (z)) - \text{ad } y\ (\text{ad } x\ (z)) \\ &= (\text{ad } x \circ \text{ad } y - \text{ad } y \circ \text{ad } x)(z) \\ &= [\text{ad } x, \text{ad } y](z). \end{aligned}$$

□

Since the map $\text{ad}\colon \mathfrak{g} \to \text{Der } \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is a homomorphism, we see that ad is a representation of $\mathfrak{g}$ on itself. For this reason, it is called the *adjoint representation* of $\mathfrak{g}$.

Let ad $\mathfrak{g}$ denote the subspace of Der $\mathfrak{g}$ consisting of all ad x, for all $x \in \mathfrak{g}$.

Proposition 3.4.3. *ad $\mathfrak{g}$ is an ideal of* Der $\mathfrak{g}$.

Proof. The proposition will follow once we prove that

$$[D, \text{ad}\, x] = \text{ad}\,(Dx). \tag{3.8}$$

for all $x \in \mathfrak{g}$ and $D \in \text{Der } \mathfrak{g}$. But for any $y \in \mathfrak{g}$, we have

$$\begin{aligned} [D, \text{ad}\, x](y) &= (D \circ \text{ad}\, x\ -\ \text{ad}\, x \circ D)(y) \\ &= D[x, y] - [x, Dy] \\ &= [Dx, y] \\ &= \text{ad}\,(Dx)\ (y), \end{aligned} \tag{3.9}$$

since D is a derivation of $\mathfrak{g}$. □

We say that $\mathfrak{g}$ is *complete* if $\text{ad}\, \mathfrak{g} = \text{Der } \mathfrak{g}$.

The kernel of the adjoint representation $x \mapsto \text{ad}\, x$ of $\mathfrak{g}$ into Der $\mathfrak{g}$ consists of all $x \in \mathfrak{g}$ such that $\text{ad}\, x = 0$; i.e., all x such that $[x, y] = 0$ for all $y \in \mathfrak{g}$. This is precisely the center $\mathfrak{c}$ of $\mathfrak{g}$.

Corollary 3.4.4. *If the center $\mathfrak{c}$ of $\mathfrak{g}$ is $\{0\}$ (e.g., when $\mathfrak{g}$ is simple), then ad maps $\mathfrak{g}$ isomorphically onto the ideal ad $\mathfrak{g}$ of* Der $\mathfrak{g}$*, so $\mathfrak{g}$ is embedded in* Der $\mathfrak{g}$.

Proof. $\mathfrak{c}$ is the kernel of the adjoint representation, so by the Corollary 3.2.5, ad maps $\mathfrak{g}$ isomorphically onto $\text{ad}\, \mathfrak{g}$. □

Proposition 3.4.5. *If* $\mathfrak{c} = \{0\}$*, then the centralizer* $\mathfrak{c}(ad\ \mathfrak{g})$ *in* $\operatorname{Der} \mathfrak{g}$ *is* $\{0\}$*, hence* $\mathfrak{c}(\operatorname{Der} \mathfrak{g}) = \{0\}$*, so* $\operatorname{Der} \mathfrak{g}$ *is embedded in* $\operatorname{Der}(\operatorname{Der} \mathfrak{g})$*.*

Proof. Suppose that $D \in \mathfrak{c}(\operatorname{ad} \mathfrak{g})$. Then for all $x \in \mathfrak{g}$, we have by by (3.9), $0 = [D, \operatorname{ad} x] = \operatorname{ad}(Dx)$. Since $\mathfrak{c} = \{0\}$, we see that $Dx = 0$ for all $x \in \mathfrak{g}$, and hence $D = 0$. Now since the elements of the center $\mathfrak{c}(\operatorname{Der} \mathfrak{g})$ kill everything in $\operatorname{Der} \mathfrak{g}$, we see that $\mathfrak{c}(\operatorname{Der} \mathfrak{g}) \subset \mathfrak{c}(\operatorname{ad} \mathfrak{g})$, whence $\mathfrak{c}(\operatorname{Der} \mathfrak{g}) = \{0\}$. By the previous corollary, this implies that $\operatorname{Der} \mathfrak{g}$ is embedded in $\operatorname{Der}(\operatorname{Der} \mathfrak{g})$. ☐

So, amusingly, if the center $\mathfrak{c}$ of $\mathfrak{g}$ is $\{0\}$, we have an increasing chain of Lie algebras

$$\mathfrak{g} \subset \operatorname{Der} \mathfrak{g} \subset \operatorname{Der}(\operatorname{Der} \mathfrak{g}) \subset \operatorname{Der}(\operatorname{Der}(\operatorname{Der} \mathfrak{g})) \subset \cdots$$

According to a theorem by Schenkman, this chain eventually stops growing. (See Schenkmann's paper [Sch51] or Jacobson's book ([Jac79] p.56, where it's an exercise) for details.)

Definition 3.4.6. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. We say that $\mathfrak{g}$ is a *direct sum* of the ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Exercise 3.4.7. If $\mathfrak{c} = \{0\}$, then show that $\mathfrak{g}$ is complete if and only if $\mathfrak{g}$ is a direct summand of any Lie algebra $\mathfrak{m}$ which contains $\mathfrak{g}$ as an ideal: $\mathfrak{m} = \mathfrak{g} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is another ideal of $\mathfrak{m}$.

Exercise 3.4.8. If $\mathfrak{g}$ is simple, show that $\operatorname{Der} \mathfrak{g}$ is complete.

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