Chapter 2

Lie Algebras: Definition and Basic Properties

2.1 Elementary Properties

Let $\mathcal{A}$ be a vector space over $\mathbb{F}$. $\mathcal{A}$ is said to be an *algebra* over $\mathbb{F}$ if there is a binary operation (a "product") on $\mathcal{A}$

$$\begin{aligned} \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (a, b) &\mapsto a.b \end{aligned}$$

such that for all $a, b, c \in \mathcal{A}$ and $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} a.(b + c) &= a.b + a.c \\ (a + b).c &= a.c + b.c \\ \alpha(a.b) &= (\alpha a).b = a.(\alpha b). \end{aligned} \tag{2.1}$$

(We say the the product is *bilinear* over $\mathbb{F}$.) The algebra $\mathcal{A}$ is *commutative*, or *abelian*, if $a.b = b.a$ for all a, $b \in \mathcal{A}$. $\mathcal{A}$ is *associative* if $a.(b.c) = (a.b).c$ for all $a, b, c \in \mathcal{A}$.

There are numerous examples of algebras. Here are but a pitiful few:

1. $\mathbb{F}[z_1, \ldots, z_n]$, the algebra of polynomials in the variables $z_, \ldots, z_n$, with coefficients in $\mathbb{F}$, is a commutative, associative algebra over $\mathbb{F}$.

2. The vector space $\mathbb{F}^{n\times n}$ of $n \times n$ matrices with entries in $\mathbb{F}$ is an associative (but not commutative) algebra over $\mathbb{F}$, of dimension n^2.

3. The vector space $\mathcal{L}(V)$ is an associative algebra under composition of linear operators. If we fix a basis B of V and identify each $T \in \mathcal{L}(V)$ with its matrix $M_B(T)$, then by (1.6), we see that $\mathcal{L}(V)$ is the same as the algebra $M_{\dim V}(\mathbb{F})$.

4. The algebra $\mathbb{H}$ of quaternions is the vector space $\mathbb{R}^4$, in which each element is written in the form $\mathbf{a} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ (with $a_0, \ldots, a_3 \in \mathbb{R}$), and in which the multiplication is defined by the following rule, extended distributively:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1 \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}. \end{aligned} \tag{2.2}$$

   Then $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$, or a 2-dimensional algebra over $\mathbb{C}$ (with basis $(1, \mathbf{j})$). It is associative but not commutative.

5. The exterior algebra ΛV of a vector space V over $\mathbb{F}$, with wedge multiplication, is an associative, noncommutative algebra over $\mathbb{F}$, of dimension $2^{\dim V}$.

6. Let G be any group. The *group algebra* $\mathbb{F}[G]$ is an associative algebra which is constructed as follows. The elements of $\mathbb{F}[G]$ are finite linear combinations of elements of G, with coefficients in $\mathbb{F}$:

$$\sum_{g \in G} a_g \, g.$$

   In the above sum, $a_g \in \mathbb{F}$ and all but a finite number of a_g are 0. Addition and scalar multiplication are defined componentwise:

$$\sum_{g \in G} a_g \, g + \sum_{g \in G} b_g \, g = \sum_{g \in G} (a_g + b_g) \, g$$

   and

$$c \cdot \sum_{g \in G} a_g \, g = \sum_{g \in G} (c a_g) \, g.$$

   Multiplication is defined so as to conform to the conditions (2.1):

$$\left(\sum_{g \in G} a_g \, g \right) \left(\sum_{h \in G} b_h \, h \right) = \sum_{g, h \in G} (a_g \, b_h) \, gh.$$

   The right hand side above can also be written as

$$\sum_{g \in G} \left(\sum_{u \in G} a_u b_{u^{-1} g} \right) g.$$

7. Let X be a topological space. The vector space $\mathbb{F}(X)$ of all continuous functions from X to $\mathbb{F}$, equipped with pointwise addition and multiplication, is a commutative, associative algebra over $\mathbb{F}$.

Definition 2.1.1. Let $\mathfrak{g}$ be an algebra over $\mathbb{F}$, with product $[x, y]$. $\mathfrak{g}$ is called a *Lie algebra* if

$$[x, x] = 0 \qquad \text{for all } x \in \mathfrak{g} \tag{2.3}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{for all } x, y, z \in \mathfrak{g}. \tag{2.4}$$

The multiplication on the Lie algebra $\mathfrak{g}$ is called the *Lie bracket*. Any algebra product which satisfies (2.3) is said to be *anticommutative*. The identity (2.4) is called the *Jacobi identity*.

Note: We have adopted the common practice of Lie theorists (those who study Lie groups and Lie algebras) to use lowercase gothic letters to denote Lie algebras, and to use $\mathfrak{g}$ to denote a typical Lie algebra. I don't know exactly how this practice came about, but I suspect that it may have had something to do with Hermann Weyl, a German who was a leading practitioner of Lie theory (and in fact one of the greatest mathematicians of the Twentieth Century) in the 1950s.

Proposition 2.1.2. *The condition* (2.3) *is equivalent to the condition that*

$$[x, y] = -[y, x]$$

for all $x, y \in \mathfrak{g}$.

Proof. Suppose that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. then $[x, x] = -[x, x] \implies [x, x] = 0$ for all $x \in \mathfrak{g}$.

Conversely, if $[x, x] = 0$ for all $x \in \mathfrak{g}$, then for all $x, y \in \mathfrak{g}$,

$$\begin{aligned} 0 &= [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] \\ &\implies [x, y] = -[y, x]. \end{aligned}$$

□

Proposition 2.1.3. *The Jacobi identity is equivalent to its alter ego:*

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \tag{2.5}$$

Proof. Easy, just multiply the Jacobi identity by -1 and use the preceding proposition. □

We now consider some examples of Lie algebras.

Example 2.1.4. An *abelian* Lie algebra is one in which the Lie bracket is commutative. Thus, in an abelian Lie algebra $\mathfrak{g}$, $[x, y] = [y, x]$, for all $x, y \in \mathfrak{g}$. But then, we also have $[x, y] = -[y, x]$, so $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. Thus the Lie bracket in an abelian Lie algebra is identically 0.

Conversely, if we define the bracket on a vector space V by $[x, y] = 0$ for all $x, y \in V$, we see that the conditions (2.3) and (2.4) are immediately satisfied. Thus, any vector space may be endowed with the (obviously trivial) structure of an abelian Lie algebra.

Exercise 2.1.5.

Example 2.1.6. Here's a great source of Lie algebras: take any associative algebra $\mathcal{A}$, and define a Lie bracket on $\mathcal{A}$ by putting

$$[x, y] := xy - yx.$$

Of course we need to verify that $[x, y]$ is indeed a Lie bracket on $\mathcal{A}$. But the anticommutativity is obvious, and the verification of the Jacobi identity is a routine calculation:

$$\begin{aligned}
&[x, [y, z]] + [y, [x, z]] + [z, [x, y]] \\
&= x(yz - zy) - (yz - zy)x + y(xz - zx) - (xz - zx)y + z(xy - yx) - (xy - yx)z \\
&= xyz - xzy - yzx + zyx + yxz - yzx - xzy + zxy + zxy - zyx - xyz + yxz \\
&= 0.
\end{aligned}$$

The Lie bracket $[x, y]$ defined above is called the *commutator product* of x and y.

The associative algebra $\mathbb{F}^n$ of $n \times n$ matrices can therefore be given a Lie algebra structure. So equipped, we will refer to this Lie algebra as $\mathfrak{gl}(n, \mathbb{F})$.

Example 2.1.7. Let V be an n-dimensional vector space over $\mathbb{F}$. As we have already seen, $\mathcal{L}(V)$ is an associative algebra of dimension n^2, and if we fix a basis B of V, $\mathcal{L}(V)$ is isomorphic (as an associative algebra) to $\mathbb{F}^n$ under the map $T \mapsto M_{B,B}(T)$. From the preceding example, $\mathcal{L}(V)$ has a Lie algebra structure, given by $[S, T] = ST - TS$. The notation $\mathfrak{gl}(V)$ will denote the associative algebra $\mathcal{L}(V)$ when it is given this Lie algebra structure

Example 2.1.8.

Definition 2.1.9. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $\mathfrak{s}$ a vector subspace of $\mathfrak{g}$. We say that $\mathfrak{s}$ is a *Lie subalgebra of* $\mathfrak{g}$ if $\mathfrak{s}$ is closed under the Lie bracket in $\mathfrak{g}$. That is, $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$ if $[x, y] \in \mathfrak{s}$ whenever x, $y \in \mathfrak{s}$.

Example 2.1.10.

Example 2.1.11. *Any* one-dimensional subspace $\mathbb{F}x$ of a Lie algebra $\mathfrak{g}$ is an abelian Lie subalgebra of $\mathfrak{g}$, for $[cx, dx] = cd[x, x] = 0$ for all c, $d \in \mathbb{F}$.

Example 2.1.12.

Definition 2.1.13. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and let $\mathfrak{s}$ be a vector subspace of $\mathfrak{g}$. We say that $\mathfrak{s}$ is an *ideal* of $\mathfrak{g}$ if $[s,x]\in\mathfrak{s}$ whenever $s\in\mathfrak{s}$ and $x\in\mathfrak{g}$.

Thus, the ideal $\mathfrak{s}$ "absorbs" elements of $\mathfrak{g}$ under the Lie bracket. An ideal $\mathfrak{s}$ of a Lie algebra $\mathfrak{g}$ is obviously a Lie subalgebra of $\mathfrak{g}$.

Example 2.1.14. Let $\mathfrak{sl}(n,\mathbb{F})$ denote the set of all $n\times n$ matrices X with entries in $\mathbb{F}$ such that $\operatorname{tr}(X)=0$. Since the trace map

$$\begin{aligned}\mathfrak{gl}(n,\mathbb{F}) &\to \mathbb{F}\\ X &\mapsto \operatorname{tr}(X)\end{aligned}$$

is a surjective linear functional on $\mathfrak{gl}(n,\mathbb{F})$, we see that $\mathfrak{sl}(n,\mathbb{F})=\ker(\operatorname{tr})$ is a vector subspace of $\mathfrak{gl}(n,\mathbb{F})$, of dimension n^2-1. We claim that $\mathfrak{sl}(n,\mathbb{F})$ is an ideal of $\mathfrak{gl}(n,\mathbb{F})$. For this, we just need to verify that $\operatorname{tr}[X,Y]=0$ whenever $X\in\mathfrak{sl}(n,\mathbb{F})$ and $Y\in\mathfrak{gl}(n,\mathbb{F})$.

But by Proposition 1.4.3, we know that for *any* X and *any* Y in $\mathfrak{gl}(n,\mathbb{F})$, we have $\operatorname{tr}[X,Y]=\operatorname{tr}(XY-YX)=\operatorname{tr}(XY)-\operatorname{tr}(YX)=0$! Thus $\mathfrak{sl}(n,\mathbb{F})$ is an ideal of $\mathfrak{gl}(n,\mathbb{F})$.

Example 2.1.15. Let V be a nonzero vector space over $\mathbb{F}$, and let $\mathfrak{sl}(V)=\{T\in\mathfrak{gl}(V)\,|\,\operatorname{tr}(T)=0\}$. Then just as in the preceding example, it is easy to prove that $\mathfrak{sl}(V)$ is an ideal of $\mathfrak{gl}(V)$.

The next proposition gives rise to a large class of the so-called classical simple Lie algebras.

Proposition 2.1.16. *Let S be a nonsingular $n\times n$ matrix over $\mathbb{F}$. Then let*

$$\mathfrak{g} := \{X\in\mathfrak{gl}(n,\mathbb{F})\,|\,S\,{}^tX\,S^{-1}=-X\} \tag{2.6}$$

Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{F})$. Moreover, $\mathfrak{g}\subset\mathfrak{sl}\,(n,\mathbb{F})$.

Proof. It is straightforward to check that $\mathfrak{g}$ is a subspace of $\mathfrak{gl}(n,\mathbb{F})$. What's important is to prove that $\mathfrak{g}$ is closed under the Lie bracket in $\mathfrak{gl}(n,\mathbb{F})$. That is, we must prove that $[X,Y]\in\mathfrak{g}$ whenever X and Y are in $\mathfrak{g}$.

But then, for X, $Y\in\mathfrak{g}$,

$$\begin{aligned}S({}^t[X,Y])S^{-1} &= S({}^t(XY-YX))\,S^{-1}\\ &= S({}^tY\,{}^tX-{}^tX\,{}^tY)\,S^{-1}\\ &= S({}^tY\,{}^tX)\,S^{-1}-S({}^tX\,{}^tY)\,S^{-1}\\ &= (S\,{}^tY\,S^{-1})(S\,{}^tX\,S^{-1})-(S\,{}^tX\,S^{-1})\,(S\,{}^tY\,S^{-1})\\ &= (-Y)(-X)-(-X)(-Y)\\ &= (YX-XY)\\ &= -[X,Y],\end{aligned}$$

which shows that $[X, Y]$ indeed belongs to $\mathfrak{g}$.

For any $X \in \mathfrak{g}$, we have $\operatorname{tr}(S\, {}^tX\, S^{-1}) = \operatorname{tr}(-X)$, which gives $\operatorname{tr} X = -\operatorname{tr} X$, and so $\operatorname{tr} X = 0$. Thus $\mathfrak{g}$ is a subalgebra of $\mathfrak{sl}(n, \mathbb{F})$. □

Example 2.1.17. When we let $S = I_n$ in Proposition 2.1.16, we obtain the Lie algebra $\mathfrak{so}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) \,|\, {}^tX = -X\}$. $\mathfrak{so}(n, \mathbb{F})$ consists of all *skew-symmetric* matrices in $\mathbb{F}$.

By convention, the real Lie algebra $\mathfrak{so}(n, \mathbb{R})$ is often simply written as $\mathfrak{so}(n)$.

Example 2.1.18.

Exercise 2.1.19. (Easy exercise.)

Example 2.1.20. Let $n = p + q$, where $p,\, q \in \mathbb{Z}^+$. If in Proposition 2.1.16, we let S be the $(p + q) \times (p + q)$ matrix which in block form is given by

$$S = I_{p,q} := \begin{pmatrix} -I_p & 0_{p\times q} \\ 0_{q\times p} & I_q \end{pmatrix}, \tag{2.7}$$

then we obtain the Lie subalgebra

$$\mathfrak{so}(p, q, \mathbb{F}) = \{X \in \mathfrak{gl}(p + q, \mathbb{F}) \,|\, I_{p,q}\, {}^tX\, I_{p,q} = -X\} \tag{2.8}$$

of $\mathfrak{sl}(p + q, \mathbb{F})$. When $\mathbb{F} = \mathbb{R}$, this Lie algebra is denoted simply by $\mathfrak{so}(p, q)$. Note that $\mathfrak{so}(p, 0, \mathbb{F}) = \mathfrak{so}(0, p, \mathbb{F}) = \mathfrak{so}(p, \mathbb{F})$.

We recall that the *adjoint*, or *transposed conjugate*, of a complex matrix X is the matrix $X^* = {}^t\overline{X}$.

Proposition 2.1.21. *Let S be a nonsingular complex $n \times n$ matrix, and let*

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \,|\, S\, X^*\, S^{-1} = -X\}. \tag{2.9}$$

Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

The easy proof, which is quite similar to that of Proposition 2.1.16, will be omitted.

Example 2.1.22. In Proposition 2.1.21, if we let $S = I_n$, we get the Lie algebra

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \,|\, X^* = -X\} \tag{2.10}$$

of *skew-Hermitian matrices.* Intersecting this with $\mathfrak{sl}(n, \mathbb{C})$, we get the Lie algebra

$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C}) \tag{2.11}$$

of skew-Hermitian matrices of trace 0. (Here we are using the easily checked fact that the intersection of two Lie subalgebras of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.)

Example 2.1.23. If $n = p + q$ and if, in Proposition 2.1.21, we let $S = I_{p,q}$, as in equation 2.7, then we obtain the Lie subalgebra

$$\mathfrak{u}(p,q) = \{X \in \mathfrak{gl}\,(p+q,\mathbb{C}) \mid I_{p,q}\,X^*\,I_{p,q} = -X\} \tag{2.12}$$

of $\mathfrak{gl}\,(p+q,\mathbb{C})$. The intersection $\mathfrak{u}(p,q) \cap \mathfrak{sl}\,(p+q,\mathbb{C})$ is denoted by $\mathrm{su}(p,q)$.

Exercise 2.1.24. Show that $\mathfrak{sl}(n,\mathbb{F})$, $\mathfrak{so}(n,\mathbb{F})$, and $\mathfrak{sp}(n,\mathbb{F})$ are all invariant under the transpose map $X \mapsto {}^tX$. Show that $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{u}(p,q)$, and $\mathfrak{su}(p,q)$ are all invariant under the adjoint map $X \mapsto X^*$.

Example 2.1.25. Suppose that $\langle\ ,\ \rangle$ is a bilinear form on a vector space V over $\mathbb{F}$. Let $\mathfrak{g}$ denote the set of all $T \in \mathcal{L}(V)$ which satisfies Leibniz' rule with respect to $\langle\ ,\ \rangle$:

$$\langle T(v), w\rangle + \langle v, T(w)\rangle = 0 \qquad\qquad \text{for all } v,\ w \in V.$$

It is easy to check that $\mathfrak{g}$ is a vector subspace of $\mathcal{L}(V)$. Let us show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ under the commutator product. Suppose that S and T are in $\mathfrak{g}$. Then for any V, $w \in V$, we have

$$\begin{aligned}
\langle (ST-TS)(v), w\rangle &= \langle ST(v), w\rangle - \langle TS(v), w\rangle \\
&= -\langle T(v), S(w)\rangle + \langle S(v), T(w)\rangle \\
&= \langle v, TS(w)\rangle - \langle v, ST(w)\rangle \\
&= -\langle v, (ST-TS)(w)\rangle.
\end{aligned}$$

This shows that $[S,T] = ST - TS \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is sometimes denoted $\mathfrak{so}\,(V)$.