LECTURE 21

The Classification of Complex Simple Lie Algebras

In the first section of this lecture we introduce the Dynkin diagram associated to a semisimple Lie algebra $\mathfrak{g}$. This is an amazingly efficient way of conveying the structure of $\mathfrak{g}$: it is a simple diagram that not only determines $\mathfrak{g}$ up to isomorphism in theory, but in practice exhibits many of the properties of $\mathfrak{g}$. The main use of Dynkin diagrams in this lecture, however, will be to provide a framework for the basic classification theorem, which says that with exactly five exceptions the Lie algebras discussed so far in these lectures are all the simple Lie algebras. To do this, in §21.2 we show how to list all diagrams that arise from semisimple Lie algebras. In §21.3 we show how to recover such a Lie algebra from the data of its diagram, completing the proof of the classification theorem. All three sections are completely elementary, though §21.3 gets a little complicated; it may be useful to read it in conjunction with §22.1, where the process described is carried out in detail for the exceptional algebra $\mathfrak{g}_2$. (Note that neither §21.3 or §22.1 is a prerequisite for §22.3, where another description of $\mathfrak{g}_2$ will be given.)

§21.1. Dynkin Diagrams Associated to Semisimple Lie Algebras

For the following, we will let $\mathfrak{g}$ be a semisimple Lie algebra; as usual, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ will be fixed throughout. As we have seen, the roots R of $\mathfrak{g}$ span a real subspace of $\mathfrak{h}^*$ on which the Killing form is positive definite. We denote this Euclidean space here by $\mathbb{E}$, and the Killing form on $\mathbb{E}$ simply by

(,) instead of $B($, $)$. The geometry of how R sits in $\mathbb{E}$ is very rigid, as indicated by the pictures we have seen for the classical Lie algebras. In this section we will classify the possible configurations, up to rotation and multiplication by a positive scalar in $\mathbb{E}$. In the next section we will see that this geometry completely determines the Lie algebra.

The following four properties of the root system are all that are needed:

(1) *R is a finite set spanning* $\mathbb{E}$.
(2) $\alpha \in R \Rightarrow -\alpha \in R$, *but* $k \cdot \alpha$ *is not in* R *if* k *is any real number other than* ± 1.
(3) *For* $\alpha \in R$, *the reflection* W_α *in the hyperplane* $\alpha^\perp$ *maps* R *to itself.*
(4) *For* $\alpha, \beta \in R$, *the real number*

$$n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

is an integer.

Except perhaps for the second part of (2), these properties have been seen in Lecture 14. For example, (4) is Corollary 14.29. Note that $n_{\beta\alpha} = \beta(H_\alpha)$, and

$$W_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha. \tag{21.1}$$

For (2), consider the representation $\mathfrak{i} = \bigoplus_k \mathfrak{g}_{k\alpha}$ of the Lie algebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$. Note that all the nonzero factors but $\mathfrak{h} = \mathfrak{g}_0$ are one dimensional. We may assume α is the smallest nonzero root that appears in the string. Now, decompose $\mathfrak{i}$ as an $\mathfrak{s}_\alpha$-module:

$$\mathfrak{i} = \mathfrak{s}_\alpha \oplus \mathfrak{i}'.$$

By the hypothesis that α is the smallest nonzero root that appears in the string, $\mathfrak{i}'$ is a representation of $\mathfrak{s}_\alpha$ having no eigenspace with eigenvalue 1 or 2 for H_α. It follows that $\mathfrak{i}'$ must be trivial, i.e., $\mathfrak{g}_{k\alpha} = (0)$ for $k \neq 0$ or ± 1.

Any set R of elements in a Euclidean space $\mathbb{E}$ satisfying conditions (1) to (4) may be called an *(abstract) root system.*

Property (4) puts very strong restrictions on the geometry of the roots. If ϑ is the angle between α and β, we have

$$n_{\beta\alpha} = 2\cos(\vartheta)\frac{\|\beta\|}{\|\alpha\|}. \tag{21.2}$$

In particular,

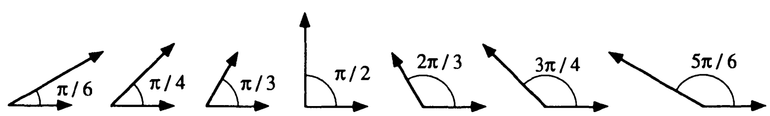
$$n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2(\vartheta) \tag{21.3}$$

is an integer between 0 and 4. The case when this integer is 4 occurs when $\cos(\vartheta) = \pm 1$, i.e. $\beta = \pm\alpha$. Omitting this trivial case, the only possibilities are therefore those given in the following table. Here we have ordered the two roots so that $\|\beta\| \geq \|\alpha\|$, or $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$.

Table 21.4

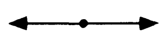
$\cos(\vartheta)$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
ϑ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\dfrac{\|\beta\|}{\|\alpha\|}$	$\sqrt{3}$	$\sqrt{2}$	1	*	1	$\sqrt{2}$	$\sqrt{3}$

In other words, the relation of any two roots α and β is one of



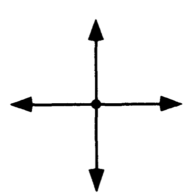
The dimension $n = \dim_{\mathbb{R}} \mathbb{E} = \dim_{\mathbb{C}} \mathfrak{h}$ is called the *rank* (of the Lie algebra, or the root system). It is easy to find all those of smallest ranks. As we write them down, we will label them by the labels (A_n), (B_n), ... that have become standard.

Rank 1. The only possibility is

(A_1) 

which is the root system of $\mathfrak{sl}_2\mathbb{C}$.

Rank 2. Note first that by Property (3), the angle between two roots must be the same for any pair of adjacent roots in a two-dimensional root system. As we will see, any of the four angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ can occur; once this angle is specified the relative lengths of the roots are determined by Property (4), except in the case of right angles. Thus, up to scalars there are exactly four root systems of dimension two. First we have the case $\vartheta = \pi/2$,

$(A_1 \times A_1)$ 

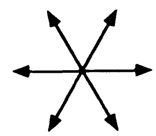
which is the root system of $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_4\mathbb{C}$.

(In general, the orthogonal direct sum of two root systems is a root system;

a root system that is not such a sum is called *irreducible*. Our task will be to classify all irreducible root systems.)

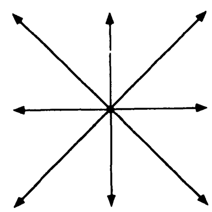
The other root systems of rank 2 are

(A_2)



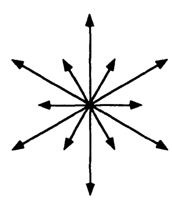
the root system of $\mathfrak{sl}_3\mathbb{C}$;

(B_2)



the root system of $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$; and

(G_2)



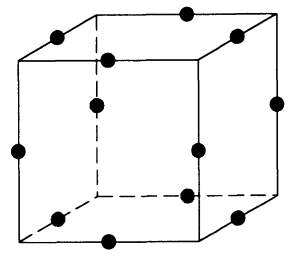
Although we have not yet seen a Lie algebra with this root system, we will see that there is one.

Exercise 21.5. Show that these are all the root systems of rank 2.

Exercise 21.6. Show that a semisimple Lie algebra is simple if and only if its root system is irreducible.

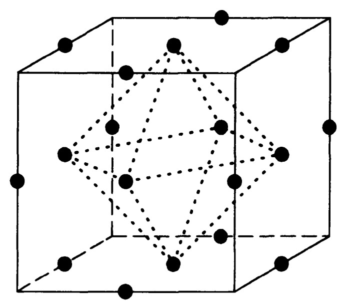
Rank 3. Besides the direct sums of (A_1) with one of those of rank 2, we have the irreducible root systems we have seen; we draw only dots at the ends of the vectors, the origins being in the centers of the reference cubes:

(A_3)



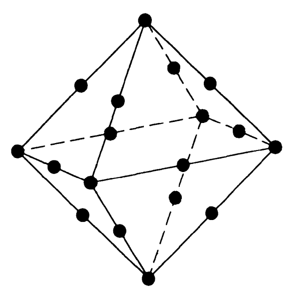
which is the root system of $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$;

(B_3)



the root system of $\mathfrak{so}_7\mathbb{C}$;

(C_3)



the root system of $\mathfrak{sp}_6\mathbb{C}$.

Exercise 21.7. Show that there are no other root systems of rank 3.

We can further reduce the data of a root system by introducing a subset of the roots, called the simple roots. First, choose as in Lecture 14 a direction

$l\colon \mathbb{E} \to \mathbb{R}$, so that $R = R^+ \cup R^-$ is a disjoint union of positive and negative roots. Call a positive root *simple* if it is not the sum of two other positive roots. For the classical Lie algebras, keeping the notations and conventions of Lectures 15–20, the simple roots are

(A_n)	$\mathfrak{sl}_{n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n - L_{n+1}$,
(B_n)	$\mathfrak{so}_{2n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n$,
(C_n)	$\mathfrak{sp}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, 2L_n$,
(D_n)	$\mathfrak{so}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n$.

Exercise 21.8. Verify this list, and find two simple roots for (G_2).

We next deduce a few consequences of properties (1)–(4), which indicate how strong these axioms are. They will be used in the present classification of abstract systems, as well as in the following section.

(5) *If α, β are roots with $\beta \neq \pm\alpha$, then the α-string through β, i.e., the roots of the form*

$$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + q\alpha$$

has at most four in a string, i.e. $p + q \leq 3$; in addition, $p - q = n_{\beta\alpha}$.

Indeed, since $W_\alpha(\beta + q\alpha) = \beta - p\alpha$, and

$$W_\alpha(\beta + q\alpha) = (\beta - n_{\beta\alpha}\alpha) - q\alpha,$$

we must have $p = n_{\beta\alpha} + q$, which is the second equality. For the first, we may take $p = 0$, and then $q = -n_{\beta\alpha}$, which we have seen is an integer no larger than three. As a consequence of (5) we have

(6) *Suppose α, β are roots with $\beta \neq \pm\alpha$. Then*

$(\beta, \alpha) > 0 \Rightarrow \alpha - \beta$ *is a root;*
$(\beta, \alpha) < 0 \Rightarrow \alpha + \beta$ *is a root.*

If $(\beta, \alpha) = 0$, then $\alpha - \beta$ and $\alpha + \beta$ are simultaneously roots or nonroots.

(7) *If α and β are distinct simple roots, then $\alpha - \beta$ and $\beta - \alpha$ are not roots.*

This follows from the definition of simple, since from the equation $\alpha = \beta + (\alpha - \beta)$, $\alpha - \beta$ cannot be in R^+, and similarly $-(\alpha - \beta) = \beta - \alpha$ cannot be in R^+. From (6) and (7) we deduce that $(\alpha, \beta) \leq 0$, i.e.,

(8) *The angle between two distinct simple roots cannot be acute.*

(9) *The simple roots are linearly independent.*

This follows from (8) by

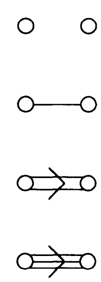
Exercise 21.9*. If a set of vectors lies on one side of a hyperplane, with all mutual angles at least 90°, show that they must be linearly independent.

(10) *There are precisely n simple roots. Each positive root can be written uniquely as a non-negative integral linear combination of simple roots.*

Since R spans $\mathbb{E}$, the first statement follows from (9), as does the uniqueness of the second statement. The fact that any positive root can be written as a positive sum of simple roots follows readily from the definition, for if α were a positive root with minimal $l(\alpha)$ that could not be so written, then α is not simple, so $\alpha = \beta + \gamma$, with β and γ positive roots with $l(\beta), l(\gamma) < l(\alpha)$.

Note that as an immediate corollary of (10) it follows that *no root is a linear combination of the simple roots α_i with coefficients of mixed sign.* For example, (7) is just a special case of this.

The *Dynkin diagram* of the root system is drawn by drawing one node ○ for each simple root and joining two nodes by a number of lines depending on the angle ϑ between them:



no lines	if $\vartheta = \pi/2$
one line	if $\vartheta = 2\pi/3$
two lines	if $\vartheta = 3\pi/4$
three lines	if $\vartheta = 5\pi/6$.

When there is one line, the roots have the same length; if two or three lines, an arrow is drawn pointing from the *longer* to the *shorter* root.

Exercise 21.10. Show that a root system is irreducible if and only if its Dynkin diagram is connected.

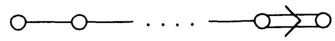
We will see later that the Dynkin diagram of a root system is independent of the choice of direction, i.e., of the decomposition of R into R^+ and R^-.

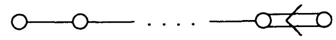
§21.2. Classifying Dynkin Diagrams

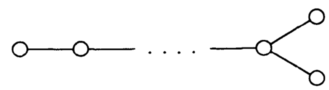
The wonderful thing about Dynkin diagrams is that from this very simple picture one can reconstruct the entire Lie algebra from which it came. We will see this in the following section; for now, we ask the complementary question of which diagrams arise from Lie algebras. Our goal is the following classification theorem, which is a result in pure Euclidean geometry. (The subscripts on the labels (A_n), ... are the number of nodes.)

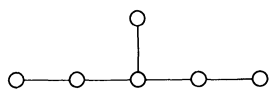
Theorem 21.11. *The Dynkin diagrams of irreducible root systems are precisely*:

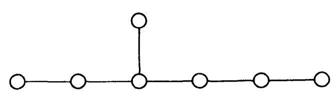
(A_n)  $(n \geq 1)$

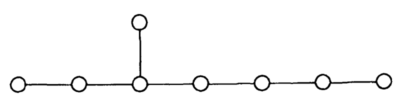
(B_n)  $(n \geq 2)$

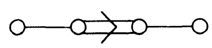
(C_n)  $(n \geq 3)$

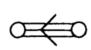
(D_n)  $(n \geq 4)$

(E_6) 

(E_7) 

(E_8) 

(F_4) 

(G_2) 

The first four are those belonging to the classical series we have been studying:

(A_n)	$\mathfrak{sl}_{n+1}\mathbb{C}$
(B_n)	$\mathfrak{so}_{2n+1}\mathbb{C}$
(C_n)	$\mathfrak{sp}_{2n}\mathbb{C}$
(D_n)	$\mathfrak{so}_{2n}\mathbb{C}$

The restrictions on n in these series are to avoid repeats, as well as degenerate cases. Indeed, the diagrams can be used to recall all the coincidences we have seen:

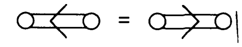
When $n = 1$, all four of the diagrams become one node. The case (D_1) is degenerate, since $\mathfrak{so}_2\mathbb{C}$ is not semisimple, while the coincidences $(C_1) = (B_1) = (A_1)$ correspond to the isomorphisms

$$\mathfrak{sp}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \qquad \circ .$$

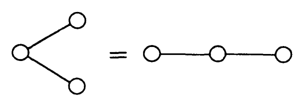
For $n = 2$, $(D_2) = (A_1) \times (A_1)$ consists of two disjoint nodes, corresponding to the isomorphism

$$\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$$  .

The coincidence $(C_2) = (B_2)$ corresponds to the isomorphism

$$\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$$  .

For $n = 3$, the fact that $(D_3) = (A_3)$ reflects the isomorphism

$$\mathfrak{so}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$$  .

Proof of the Theorem. Our desert-island reader would find this a pleasant pastime. For example, if there are two simple roots with angle $5\pi/6$, the plane of these roots must contain the G_2 configuration of 12 roots. It is not hard to see that one cannot add another root that is not perpendicular to this plane, without some of the 12 angles and lengths being wrong. This shows that (G_2) is the only connected diagram containing a triple line. At the risk of spoiling your fun, we give the general proof of a slightly stronger result.

In fact, the angles alone determine the possible diagrams. Such diagrams, without the arrows to indicate relative lengths, are often called *Coxeter diagrams* (or Coxeter graphs). Define a diagram of n nodes, with each pair connected by 0, 1, 2, or 3 lines, to be *admissible* if there are n independent unit vectors $e_1, \ldots, e_n$ in a Euclidean space $\mathbb{E}$ with the angle between e_i and e_j being $\pi/2$, $2\pi/3$, $3\pi/4$, or $5\pi/6$, according as the number of lines between corresponding nodes is 0, 1, 2, or 3. The claim is that the diagrams of the above Dynkin diagrams, ignoring the arrows, are the only connected admissible diagrams. Note that

$$(e_i, e_j) = 0, \ -1/2, \ -\sqrt{2}/2, \text{ or } -\sqrt{3}/2, \tag{21.12}$$

according as the number of lines between them is 0, 1, 2, or 3; equivalently,

$$4(e_i, e_j)^2 = \text{number of lines between } e_i \text{ and } e_j. \tag{21.13}$$

The steps of the proof are as follows:

(i) *Any subdiagram of an admissible diagram, obtained by removing some nodes and all lines to them, will also be admissible.*

(ii) *There are at most $n - 1$ pairs of nodes that are connected by lines. The diagram has no cycles (loops).*

Indeed, if e_i and e_j are connected, $2(e_i, e_j) \le -1$, and

$$0 < (\sum e_i, \sum e_i) = n + 2 \sum_{i<j} (e_i, e_j),$$

which proves the first statement of (ii). The second follows from the first and (i).

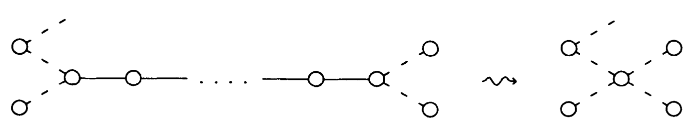
(iii) *No node has more than three lines to it.*

By (i), we may assume that e_1 is connected to each of the other nodes; by (ii), no other nodes are connected to each other. We must show that $\sum_{j=2}^{n} 4(e_1, e_j)^2 < 4$. Since $e_2, \ldots, e_n$ are perpendicular unit vectors, and e_1 is not in their span,

$$1 = (e_1, e_1)^2 > \sum_{j=2}^{n} (e_1, e_j)^2,$$

as required.

(iv) *In an admissible diagram, any string of nodes connected to each other by one line, with none but the ends of the string connected to any other nodes, can be collapsed to one node, and resulting diagram remains admissible*:

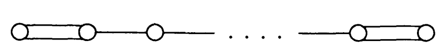


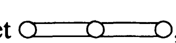
If $e_1, \ldots, e_r$ are the unit vectors corresponding to the string of nodes, then $e' = e_1 + \cdots + e_r$ is a unit vector, since

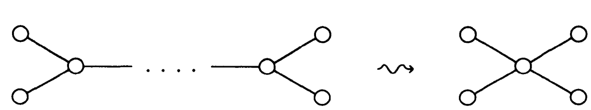
$$\begin{aligned}(e', e') &= r + 2((e_1, e_2) + (e_2, e_3) + \cdots + (e_{r-1}, e_r)) \\ &= r - (r - 1).\end{aligned}$$

Moreover, e' satisfies the same conditions with respect to the other vectors since (e', e_j) is either (e_1, e_j) or (e_r, e_j).

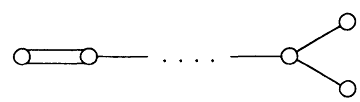
Now we can rule out the other admissible connected diagrams not on our list. First, from (iii) we see that the diagram (G_2) has the only triple edge. Next, there cannot be two double lines, or we could find a subdiagram of the form:



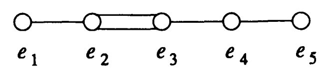
and then collapse the middle to get , contradicting (iii). Similarly there can be at most one triple node, i.e., a node with single lines to three other nodes, by



By the same reasoning, there cannot be a triple node together with a double line:



To finish the case with double lines, we must simply verify that



is not admissible. Consider general vectors $v = a_1 e_1 + a_2 e_2$, and $w = a_3 e_3 + a_4 e_4 + a_5 e_5$. We have

$$\|v\|^2 = a_1^2 + a_2^2 - a_1 a_2, \qquad \|w\|^2 = a_3^2 + a_4^2 + a_5^2 - a_3 a_4 - a_4 a_5,$$

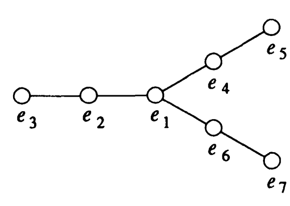
and $(v, w) = -a_2 a_3/\sqrt{2}$. We want to choose v and w to contradict the Cauchy–Schwarz inequality $(v, w)^2 < \|v\|^2\|w\|^2$. For this we want $|a_2|/\|v\|$ and $|a_3|/\|w\|$ to be as large as possible.

Exercise 21.14. Show that these maxima are achieved by taking $a_2 = 2a_1$ and $a_3 = 3a_5$, $a_4 = 2a_5$.

In fact, $v = e_1 + 2e_2$, $w = 3e_3 + 2e_4 + e_5$ do give the contradictory

$$(v, w)^2 = 18, \quad \|v\|^2 = 3, \quad \text{and} \quad \|w\|^2 = 6.$$

Finally, we must show that the strings coming out from a triple node cannot be longer that those specified in types (D_n), (E_6), (E_7), or (E_8). First, we rule out



Consider the three perpendicular unit vectors:

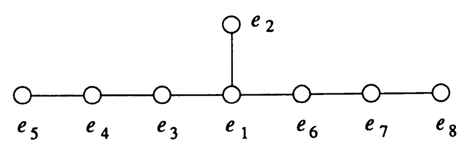
$$u = (2e_2 + e_3)/\sqrt{3}, \qquad v = (2e_4 + e_5)/\sqrt{3}, \qquad w = (2e_6 + e_7)/\sqrt{3}.$$

Then as in (iii), since e_1 is not in the span of them,

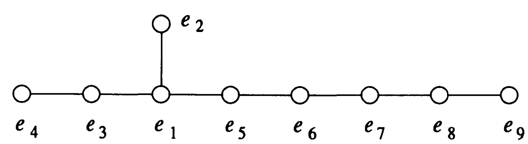
$$1 = \|e_1\|^2 > (e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/3 + 1/3 + 1/3 = 1,$$

a contradiction.

Exercise 21.15*. Similarly, rule out



and



(The last few arguments can be amalgamated, by showing that if the legs from a triple node have lengths p, q, and r, then $1/p + 1/q + 1/r$ must be greater than 1.)

This finishes the proof of the theorem. □

§21.3. Recovering a Lie Algebra from Its Dynkin Diagram

In this section we will complete the classification theorem for simple Lie algebras by showing how one may recover a simple Lie algebra from the data of its Dynkin diagram. This will proceed in two stages: first, we will see how to reconstruct a root system from its Dynkin diagram (which a priori only tells us the configuration of the simple roots). Secondly, we will show how to describe the entire Lie algebra in terms of its root system. (In the next lecture we will do all this explicitly, by hand, and independently of the general discussion here, for the simplest exceptional case (G_2); as we have noted, the reader may find it useful to work through §22.1 before or while reading the general story described here.)

To begin with, to recover the root system from the Dynkin diagram, let $\alpha_1, \ldots, \alpha_n$ be the simple roots corresponding to the nodes of a connected Dynkin diagram. We must show which non-negative integral linear combinations $\sum m_i\alpha_i$ are roots. Call $\sum m_i$ the *level* of $\sum m_i\alpha_i$. Those of level one are the simple roots. For level two, we see from Property (2) that no $2\alpha_i$ is a root, and by

Property (6) that $\alpha_i + \alpha_j$ is a root precisely when $(\alpha_i, \alpha_j) < 0$, i.e., when the corresponding nodes are joined by a line.

Suppose we know all positive roots of level at most m, and let $\beta = \sum m_i\alpha_i$ be any positive root of level m. We next determine for each simple root $\alpha = \alpha_j$, whether $\beta + \alpha$ is also a root. Look at the α-string through β:

$$\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha.$$

We know p by induction (no root is a linear combination of the simple roots α_i with coefficients of mixed sign, so $p \leq m_j$ and $\beta - p\alpha$ is a positive root). By Property (5), $q = p - n_{\beta\alpha}$. So $\beta + \alpha$ is a root exactly when

$$p > n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = \sum_{i=1}^{n} m_i n_{\alpha_i\alpha}.$$

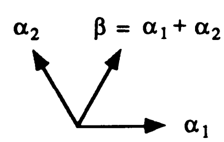
In effect, the additional roots we will find in this way are those obtained by reflecting a known positive root in the hyperplane perpendicular to a simple root α_i (and filling in the string if necessary).

To finish the proof, we must show that we get all the positive roots in this way. This will follow once from the fact that any positive root of level $m + 1$ can be written in at least one way as a sum of a positive root of level m and a simple root. If $\gamma = \sum r_i\alpha_i$ has level $m + 1$, from

$$0 < (\gamma, \gamma) = \sum r_i(\gamma, \alpha_i),$$

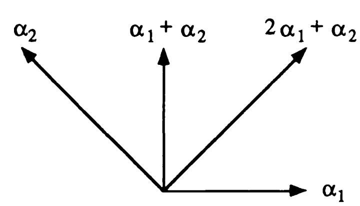
some (γ, α_i) must be positive, with $r_i > 0$. By property (6), $\gamma - \alpha_i$ is a root, as required.

By way of example, consider the rank 2 root systems. In the case of $\mathfrak{sl}_3\mathbb{C}$, we start with a pair of simple roots α_1, α_2 with $n_{\alpha_1,\alpha_2} = -1$, i.e., at an angle of $2\pi/3$; as always, we know that $\beta = \alpha_1 + \alpha_2$ is a root as well.



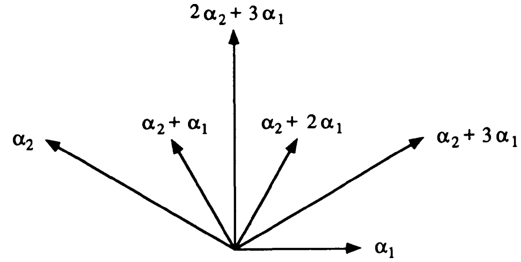
On the other hand, since $\beta - 2\alpha_1 = \alpha_2 - \alpha_1$ is not a root, $\beta + \alpha_1$ cannot be either, and likewise $\beta + \alpha_2$ is not; so we have all the positive roots.

In the case of $\mathfrak{sp}_4\mathbb{C}$, we have two simple roots α_1 and α_2 at an angle of $3\pi/4$; in terms of an orthonormal basis L_1 and L_2 these may be taken to be L_1 and $L_2 - L_1$, respectively.



We then see that in addition to $\beta = \alpha_1 + \alpha_2$, the sum $\beta + \alpha_1 = 2\alpha_1 + \alpha_2$ is a root—it is just the reflection of α_2 in the plane perpendicular to α_1—but $\beta + \alpha_2 = \alpha_1 + 2\alpha_2$ and $3\alpha_1 + \alpha_2$ are not because $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_1$ are not respectively (alternatively, we could note that they would form inadmissible angles with α_1 and α_2 respectively).

Finally, in the case of (G_2), we have two simple roots α_1, α_2 at an angle of $5\pi/6$, which in terms of an orthonormal basis for $\mathbb{E}$ may be taken to be L_1 and $(-3L_1 + \sqrt{3}L_2)/2$ respectively.



Reflecting α_2 in the plane perpendicular to α_1 yields a string of roots $\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$ and $\alpha_2 + 3\alpha_2$. Moreover, reflecting the last of these in the plane perpendicular to α_2 yields one more root, $2\alpha_2 + 3\alpha_3$. Finally, these are all the positive roots, giving us the root system for the diagram (G_2).

We state here the results of applying this process to the exceptional diagrams (F_4), (E_6), (E_7), and (E_8) (in addition to (G_2)). In each case, $L_1, \ldots, L_n$ is an orthogonal basis for $\mathbb{E}$, the simple roots α_i can be taken to be as follows, and the corresponding root systems are given:

(G_2)
$$\alpha_1 = L_1, \qquad \alpha_2 = -\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2;$$

$$R^+ = \left\{L_1, \sqrt{3}L_2, \pm L_1 + \frac{\sqrt{3}}{2}L_2, \pm\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2\right\}.$$

(G_2) thus has 6 positive roots.

(F_4)
$$\alpha_1 = L_2 - L_3, \qquad \alpha_2 = L_3 - L_4, \qquad \alpha_3 = L_4,$$
$$\alpha_4 = \frac{L_1 - L_2 - L_3 - L_4}{2};$$

$$R^+ = \{L_i\} \cup \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \left\{\frac{L_1 \pm L_2 \pm L_3 \pm L_4}{2}\right\}.$$

In particular, (F_4) has 24 positive roots.

(E_6)
$$\alpha_1 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \qquad \alpha_2 = L_1 + L_2,$$

$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2,$$
$$\alpha_5 = L_4 - L_3, \qquad \alpha_6 = L_5 - L_4;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 5} \cup \{L_i - L_j\}_{j<i\le 5} \cup \left\{\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 \pm L_5 + \sqrt{3}L_6}{2}\right\}_{\text{number of minus signs even}}.$$

(E_6) has 36 positive roots.

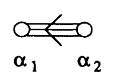
$$(E_7) \qquad \alpha_1 = \frac{L_1 - L_2 - \cdots - L_6 + \sqrt{2}L_7}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 6} \cup \{L_i - L_j\}_{j<i\le 6} \cup \{\sqrt{2}L_7\} \cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_6 + \sqrt{2}L_7}{2}\right\}_{\text{number of minus signs odd}}.$$

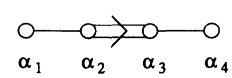
Thus, (E_7) has 63 positive roots.

$$(E_8) \qquad \alpha_1 = \frac{L_1 - L_2 - \cdots - L_7 + L_8}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5, \qquad \alpha_8 = L_7 - L_6.$$
$$R^+ = \{L_i + L_j\}_{i<j\le 8} \cup \{L_i - L_j\}_{j<i\le 8} \cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_7 + L_8}{2}\right\}_{\text{number of minus signs even}}.$$

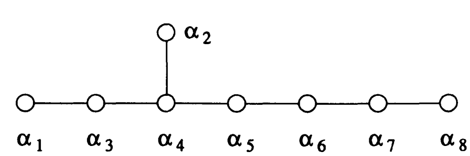
(E_8) has 120 positive roots.

For (G_2) and (F_4) the simple roots are listed in order reading from left to right in their Dynkin diagrams





as in the classical series (A_n)–(D_n). For (E_8), the numbering is



while those for (E_7) and (E_6) are obtained by removing the last one or two nodes. Note that, given the root system of (E_8), we can find the root system of (E_7) or (E_6) by taking the subspace spanned by the first seven or six simple roots.

Exercise 21.16*. (a) Verify the above lists of roots.
(b) In each case, calculate the corresponding fundamental weights.

Exercise 21.17*. Show that no two of the root systems of (A_n)–(E_8) are isomorphic, and deduce that the Dynkin diagram of a root system is independent of choice of positive roots.

A more satisfying reason for the last fact is the observation that any two choices of positive roots differ by an element of the Weyl group—the group generated by reflections W_α in the simple roots. This can be seen directly for each of the diagrams (A_n)–(E_8); for a general proof that two choices differ by an element of the Weyl group, see Proposition D.29.

We should mention here another way of conveying the data of a Dynkin diagram. This is simply the $n \times n$ matrix of integers ($n_{i,j} = n_{\alpha_i \alpha_j}$), where we take $n_{i,i} = 2$; it is called the *Cartan matrix* of the Dynkin diagram (or of the Lie algebra). Thus, for example, the Cartan matrix of (A_n) is

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}.$$

These matrices pop up remarkably often, in a variety of seemingly unrelated areas of mathematics. They will not play a major role in the present text, but the reader has probably encountered them already in one form or another, and will probably do so again.

Exercise 21.18*. Compute the Cartan matrix, and its determinant, for each Dynkin diagram.

The next task is to see how the root system determines the Lie algebra. We concentrate on the uniqueness, since there are other ways to see the existence; indeed, for all but the five exceptions we have already seen the Lie algebras. We will describe several approaches to this problem, starting with a straightforward and computational method and finishing with a slick but abstract approach.

Assume as before that $\mathfrak{g}$ is a simple Lie algebra, with a chosen Cartan subalgebra $\mathfrak{h}$ and decomposition of the roots R into positive and negative roots; let $\alpha_1, \ldots, \alpha_n$ be the simple roots. The Dynkin diagram information is the knowledge of (α_i, α_j) for all $i \neq j$. Let $H_i = H_{\alpha_i}$ be the corresponding basis of $\mathfrak{h}$, defined by the rule we have seen in Lecture 14: if $\{T_i\}$ is the basis corresponding via the Killing form to $\{\alpha_i\}$, set $H_i = 2T_i/(\alpha_i, \alpha_i)$.

Choose any nonzero element X_i in the root space $\mathfrak{g}_{\alpha_i}$, for $1 \leq i \leq n$. This determines elements Y_i in $\mathfrak{g}_{-\alpha_i}$ such that $[X_i, Y_i] = H_i$. We claim first that these $3n$ elements $\{H_i, X_i, Y_i\}$ generate $\mathfrak{g}$ as a Lie algebra. This follows from

Claim 21.19. *If α, β, and $\alpha + \beta$ are roots, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. Again look at the α-string through $\mathfrak{g}_\beta$, i.e., $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$. This is an irreducible representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$, since all the terms are one dimensional (this follows from the fact that no $\beta + k\alpha$ can be zero, given that $\beta \neq \pm\alpha$). But now if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, $\bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ would be a nontrivial subrepresentation. □

For each positive root β, we have seen that can write β as a sum of simple roots $\beta = \alpha_{i_1} + \cdots + \alpha_{i_r}$ such that each of the sums $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \leq s \leq r$. If we choose such a presentation for each β, and set

$$X_\beta = [X_{i_r}, [X_{i_{r-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots]]$$

and

$$Y_\beta = [Y_{i_r}, [Y_{i_{r-1}}, \ldots, [Y_{i_2}, Y_{i_1}] \ldots]],$$

then the collection

$$\{H_i, 1 \leq i \leq n; X_\beta, Y_\beta, \beta \in R^+\} \tag{21.20}$$

forms a basis for $\mathfrak{g}$. Note that if β is not simple, there is no reason to expect $[X_\beta, Y_\beta]$ to be the distinguished element H_β in $\mathfrak{h}$.

We want to show that the multiplication table for these basis elements is completely determined by the Dynkin diagram. The main difficulty is that the ordering of the simple roots in the above expression for β may not be unique. For example, suppose

$$\beta = (\alpha_1 + \alpha_2) + \alpha_3 = (\alpha_2 + \alpha_3) + \alpha_1,$$

with $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ roots. We must compare $[X_3, [X_2, X_1]]$ with $[X_1, [X_3, X_2]]$. In fact, they must be negatives of each other. For, by Jacobi, we have

$$[X_1, [X_3, X_2]] = -[X_3, [X_2, X_1]] - [X_2, [X_1, X_3]] = -[X_3, [X_2, X_1]],$$

noting that $[X_1, X_3] = 0$ since $\alpha_1 + \alpha_3$ cannot be a root, e.g., by step (ii) of the preceding section.

For any sequence $I = (i_1, \ldots, i_r)$, $1 \leq i_j \leq n$, set

$$\alpha_I = \alpha_{i_1} + \cdots + \alpha_{i_r},$$
$$X_I = [X_{i_r}, [X_{i_{r-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots]],$$
$$Y_I = [Y_{i_r}, [Y_{i_{r-1}}, \ldots, [Y_{i_2}, Y_{i_1}] \ldots]].$$

Call I *admissible* if each partial sum $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \le s \le r$; note that I is admissible exactly when X_I is not zero.

Lemma 21.21. *If I and J are two admissible sequences for which $\alpha_I = \alpha_J$, then there is a nonzero rational number q determined by I, J, and the Dynkin diagram, such that $X_J = q \cdot X_I$.*

PROOF. Let $k = i_r$ be the last entry in I. If $j_r = k$ as well, the result follows by induction on r. We reduce the general case to this case, by maneuvering to replace j_r by k. We have first

$$X_J = q_1 \cdot [X_k, [Y_k, X_J]],$$

with q_1 a nonzero rational number depending only on J, k, and the Dynkin diagram, since $\alpha_J - \alpha_k = \alpha_I - \alpha_k$ is a root; the point is that we know how $\mathfrak{s}_{\alpha_k} \cong \mathfrak{sl}_2$ acts on the α_k-string through α_J as soon as we know the length of the string, and this is Dynkin diagram information. Next, let s be the largest integer such that $j_s = k$. Then

$$[Y_k, X_J] = [X_{j_r}, \ldots [X_{j_{s+1}}, [Y_k, [X_k, X_K]]] \ldots],$$

where $K = (j_1, \ldots, j_{s-1})$, since $[Y_k, [X_i, Z]] = [X_i, [Y_k, Z]]$ when $i \neq k$. Finally,

$$[Y_k, [X_k, X_K]] = q_2 \cdot X_K,$$

with q_2 a nonzero rational number depending only on K, k, and the Dynkin diagram, since $\alpha_K + \alpha_k$ is a root. Combining these three equations, we get

$$X_J = q_1 q_2 \cdot [X_k, [X_{j_r}, \ldots [X_{j_{s+1}}, X_K] \ldots]],$$

which suffices since the sequence for the term on the right ends in the same integer k as I. □

Proposition 21.22. *The bracket of any two basis elements in* (21.20) *is a rational multiple of another basis element, that multiple determined from the Dynkin diagram.*

PROOF. This is clear for brackets of an H_i with any basis element. Lemma 21.21 handles brackets of the form $[X_I, X_J]$, and those involving only Y's are similar. For brackets $[Y_I, X_J]$, it suffices inductively to compute $[Y_k, X_J]$ as a rational multiple of some X_K, with K shorter than J (or of H_k if J has one term); but this was worked out in the proof of the lemma. □

Exercise 21.23*. (i) Show that in (G_2) each positive root can be written in only one way as a sum of simple roots, up to the order of the first two roots.

(ii) Work out the multiplication table from the Dynkin diagram. (iii) Verify that the result is indeed a Lie algebra, which is (visibly) simple.

This exercise will be worked out in detail to start the next lecture. Of course, there is nothing but lack of time to keep us from verifying that the other four exceptional Dynkin diagrams do lead, by the same prescription, to honest Lie algebras, but doing it by hand gets pretty laborious, and we will describe some of the other methods available.

The fact that the multiplication table can be defined with rational coefficients becomes important when one wants to reduce them modulo prime numbers, which we will not discuss here. The fact that they can be taken to be real, on the other hand, will come up later, when we discuss real forms of complex Lie algebras and groups.

There is a more general and elegant way to proceed, given by Serre [Se3]. Write n_{ij} in place of $n_{\alpha_i\alpha_j}$. Form the free Lie algebra on generators

$$H_1, \ldots, H_n, X_1, \ldots, X_n, Y_1, \ldots, Y_n,$$

i.e., form the free (tensor) algebra with this basis, and divide modulo by the relations $[A, B] + [B, A] = 0$ and the Jacobi relation. Then take this free Lie algebra, and divide by the relations

$$[H_i, H_j] = 0 \text{ (all } i, j); \qquad [X_i, Y_i] = H_i \text{ (all } i); \qquad [X_i, Y_j] = 0 \ (i \neq j);$$

$$[H_i, X_j] = n_{ji} X_j \text{ (all } i, j); \qquad [H_i, Y_j] = -n_{ji} Y_j \text{ (all } i, j);$$

and, for all $i \neq j$,

$$\begin{aligned}
[X_i, X_j] &= 0, & [Y_i, Y_j] &= 0 & &\text{if } n_{ij} = 0;\\
[X_i, [X_i, X_j]] &= 0, & [Y_i, [Y_i, Y_j]] &= 0 & &\text{if } n_{ij} = -1;\\
[X_i, [X_i, [X_i, X_j]]] &= 0, & [Y_i, [Y_i, [Y_i, Y_j]]] &= 0 & &\text{if } n_{ij} = -2;\\
[X_i, [X_i, [X_i, [X_i, X_j]]]] &= 0, & [Y_i, [Y_i, [Y_i, [Y_i, Y_j]]]] &= 0 & &\text{if } n_{ij} = -3.
\end{aligned}$$

Exercise 21.24. Verify that if one starts with a semisimple Lie algebra with a given Dynkin diagram, the above equations must hold.

Serre shows ([Se3, Chap. VI App.], cf. [Hu1 §18]) that the resulting Lie algebra is a finite-dimensional semisimple Lie algebra, with Cartan subalgebra generated by $H_1, \ldots, H_n$ and given root system. In particular, this includes a proof of the existence of all the simple Lie algebras.

Here is a third approach to uniqueness. Suppose $\mathfrak{g}$ and $\mathfrak{g}'$, with given Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, and choice of positive roots, have isomorphic root systems. There is an isomorphism $\mathfrak{h} \to \mathfrak{h}'$, taking corresponding H_i to H_i'. Choose arbitrarily nonzero vectors X_i and X_i' in the root spaces of $\mathfrak{g}$ and $\mathfrak{g}'$ corresponding to the simple roots.

Claim 21.25. *There is a unique isomorphism from* $\mathfrak{g}$ *to* $\mathfrak{g}'$ *extending the isomorphism of* $\mathfrak{h}$ *with* $\mathfrak{h}'$, *and mapping* X_i *to* X_i' *for all i.*

Proof. The uniqueness of the isomorphism is easy: the resulting map is determined on the Y_i by $\mathfrak{sl}_2$ considerations, and the H_i, X_i, and Y_i generate $\mathfrak{g}$. For the existence of the isomorphism consider the subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} \oplus \mathfrak{g}'$ generated by $\tilde{H}_i = H_i \oplus H_i'$, $\tilde{X}_i = X_i \oplus X_i'$, and $\tilde{Y}_i = Y_i \oplus Y_i'$. It suffices to prove that the two projections from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphisms. The kernel of the second projection is $\mathfrak{k} \oplus 0$, where $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{k}$ is either 0, as required, or $\mathfrak{k} = \mathfrak{g}$. In the latter case, we must have $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$.

To see that this is impossible, consider a maximal positive root β, take nonzero vectors X_β, X_β' in the corresponding root spaces, and set $\tilde{X}_\beta = X_\beta \oplus X_\beta'$, a highest weight vector in $\tilde{\mathfrak{g}}$. Let W be the subspace of $\tilde{\mathfrak{g}}$ obtained by successively applying all $\tilde{Y}_i$'s. Then W is a proper subspace of $\tilde{\mathfrak{g}}$, since its weight space W_β corresponding to β is one dimensional. By the argument we have seen several times, $\tilde{\mathfrak{g}}$ preserves W. Now if $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$, W would be an ideal in $\mathfrak{g} \oplus \mathfrak{g}'$, and this would force $X_\beta \oplus 0$ to belong to W, making W_β two dimensional again. □

To finish this story, we should show that the simple Lie algebras corresponding to two different Dynkin diagrams cannot be isomorphic, i.e., that the two choices made in going from a semisimple Lie algebra to Dynkin diagram do not change the answer. The general facts are:

(1) Any two Cartan subalgebras of a semisimple Lie algebra are conjugate, i.e., there is an inner automorphism by an element in the corresponding adjoint group, which takes one into the other.
(2) Any two decompositions of a root system into positive and negative roots differ by an element of the Weyl group.

These are standard facts which are proved in Appendix D. Both statements are subsumed in the fact that any two *Borel subalgebras* of a semisimple Lie algebra are conjugate, a Borel subalgebra being the subspace spanned by the Cartan subalgebra and the root spaces $\mathfrak{g}_\alpha$ for positive α. For those readers who crave logical completeness but do not want to go through so much general theory, we observe that most possible coincidences can be ruled out by such simple considerations as computing dimensions, and others can be ruled out by simple ad hoc methods, cf. Exercise 21.17.

Finally, we must also prove the "existence theorem": that there is a simple Lie algebra for each Dynkin diagram. Serre's theorem quoted above gives a unified proof of existence. But we have seen and studied the Lie algebras for the classical cases (A_n)–(D_n), and it is more in keeping with the spirit of these lectures to at least try to see the five exceptions explicitly. This is the subject of the next lecture.