Elements of Logic

Math 150:
To Infinity and Beyond

February, 2022

First, a Joke...

Three logicians had a nice meal at their favorite restaurant.The waiter comes by after dessert and asks:

Will you all take coffee?

- First logician: *I don't know.*
- Second logician: *I don't know.*
- Third logician: *Yes.*

Raucous laughter ensues.

An Example

How did we prove that $\sqrt{2}$ is irrational?

1. We assumed that $\sqrt{2}$ *was* in fact a fraction.
2. We said that the fraction $\frac{a}{b}$ could be chosen *irreducible*.
3. We manipulated $\frac{a}{b}$ according to the rules of basic arithmetic.
4. It appeared that $\frac{a}{b}$ could in fact not be irreducible...

How is that conclusive?

Why should we accept the conclusion?

Let's try to challenge the argument.

- Attack on (3): maybe we did a calculation mistake...
- Attack on (2): maybe some fractions cannot be simplified to an irreducible form...
- Attack on (4): maybe a fraction can be irreducible and reducible at the same time...

True of False?

The Liar's Paradox

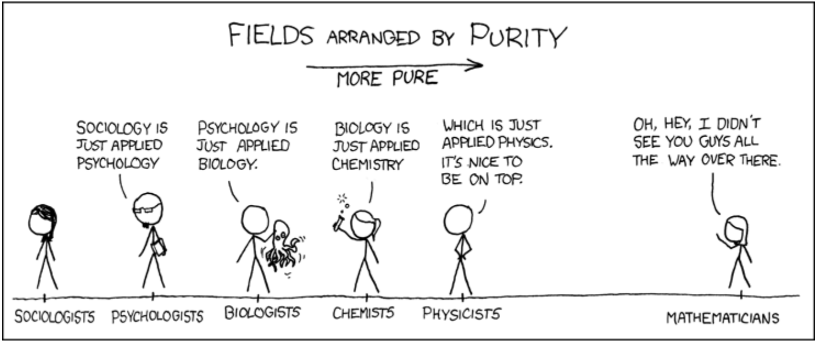
This sentence is false.

What is Logic?

For us in this course, and many others who have practical, everyday use for it, logic is the art of proper thinking.

Another way to think about it: Logic does for Mathematics what Mathematics does for Physics, or as François Dorais puts it:

> Logic is the math of math.


FIELDS ARRANGED BY PURITY
MORE PURE
SOCIOLOGY IS JUST APPLIED PSYCHOLOGY
PSYCHOLOGY IS JUST APPLIED BIOLOGY.
BIOLOGY IS JUST APPLIED CHEMISTRY
WHICH IS JUST APPLIED PHYSICS. IT'S NICE TO BE ON TOP.
OH, HEY, I DIDN'T SEE YOU GUYS ALL THE WAY OVER THERE.
SOCIOLOGISTS
PSYCHOLOGISTS
BIOLOGISTS
CHEMISTS
PHYSICISTS
MATHEMATICIANS

Generalities about Logic

Logic has a long and eventful history (from Antiquity to the 20th century), traversing many fields of study, including:

- Philosophy
- Mathematics
- Theology
- Psychology
- Cognitive Sciences...

Some (ancient) History

Etymology

The Greek word *Logos* means:

word or reason.

Major schools in Antiquity:

- Aristotelian Logic: work of Aristotle (384BC - 322BC), known through Andronicus of Rhodes (~70BC)
- Stoic Logic: work of Chrysippus (279BC - 206BC)

Aristotelian logic had an immense impact on western thinking for centuries.

Kant (1724 - 1804) believed that Aristotle had "finished" logic...

Crisis of Foundations

In the late 19th century, the methods of Mathematics had become more rigorous and mathematicians were trying to establish solid foundations for their work (Frege, Peirce...)

Some tried to gauge how far Mathematical knowledge could go.

David Hilbert believed there was no *ignorabimus* Mathematics.

Wir müssen wissen, wir werden wissen.

Hilbert, 1930.

Crisis of Foundations

Then, along came Kurt Gödel...

Gödel's Incompleteness Theorem (1930)

If a system of axioms is sophisticated enough to contain basic arithmetic, then:

- it contains undecidable propositions,
- the consistency of the axioms cannot be proved within the system.

Read all about it in *Logicomix*!

The arts of discourse in Aristotle's time

- Grammar: the proper use of the rules of language.
- Rhetoric: the art of convincing.
- Logic: the art of making proper arguments.

A learned man knows that two negatives amount to a positive.
*A wise man knows that Wu-Tang Clan ain't nuthin to f*ck with.*

What are the objects of logic?

The example of the Liar's Paradox suggests that the domain of application of logic should be limited...

Pragmatic restriction

An *assertion* is a sentence that can meaningfully be called true or false.

In particular, we shall stay away from self-referential statements.

Structure of assertions

Assertions are made of *terms*, that are connected together.

Joaquin Phoenix is a great actor.

In this assertion,

- 'Joaquin Phoenix' $\rightsquigarrow$ *the subject*
- 'a great actor' $\rightsquigarrow$ *the predicate*
- 'is' $\rightsquigarrow$ the copula.

Categorical propositions

According to *Encyclopædia Britannica*, a *categorical proposition* is:

> *A proposition or statement, in which the predicate is, without qualification, affirmed or denied of all or part of the subject.*

Categorical propositions fall in four categories:

	Quantifier	Subject	Copula	Predicate	Type
a	All	S	are	P	Universal affirmative
e	No	S	are	P	Universal negative
i	Some	S	are	P	Particular affirmative
o	Some	S	are not	P	Particular negative

Code: affirmo , nego.

All X are Y: XaY
Some X are Y: XiY

Examples

Can the statements below be reformulated as categorical propositions?

- *‘ Mathematics is the most beautiful and most powerful creation of the human spirit.’*

  – Stefan Banach (1892-1945)

- *The Wu-Tang Clan is not something to mess with.*

  – Wu-Tang Clan

- *Daniel Day-Lewis is a better actor than Joaquin Phoenix.*

Classification of categorical propositions

	Quantifier	Subject	Copula	Predicate	Type
a	All	S	are	P	Universal affirmative
e	No	S	are	P	Universal negative
i	Some	S	are	P	Particular affirmative
o	Some	S	are not	P	Particular negative

This allows to encode categorical propositions (X and Y are terms):

- XaY $\rightsquigarrow$ All X is Y.
- XeY $\rightsquigarrow$ No X is Y.
- XiY $\rightsquigarrow$ Some X is Y.
- XoY $\rightsquigarrow$ Some X are not Y.

Combining propositions: syllogisms

A famous argument:

All men are mortal.

Socrates is a man.

Therefore,

Socrates is mortal.

This is a *syllogism*. It consists of

- A major premise
- A minor premise
- A conclusion.

About the order of premises

Socrates is a man.

All men are mortal.

Therefore,

Socrates is mortal.

What changes if we switch the major and minor premise? *Nothing!*

Convention

We say that a syllogism is in *standard form* if the first premise contains the predicate of the conclusion.

Possible figures

A syllogism in standard form necessarily has one of the *figures* below:

	Figure 1	Figure 2	Figure 3	Figure 4
Major Premise	M - P	P - M	M - P	P - M
Minor Premise	S - M	S - M	M - S	M - S
Conclusion	S - P	S - P	S - P	S - P

where

- S is the subject of the conclusion
- P is the predicate of the conclusion
- M is the *middle term*.

An example

1. What is the figure of this syllogism?

Some small birds live on honey.

All birds that live on honey are colorful birds.

Therefore,

Some colorful birds are small birds.

2. Encode each proposition in this syllogism.

Classification of syllogisms

How many different types of syllogisms can there be?

Figure 1	**Figure 2**	**Figure 3**	**Figure 4**
Barbara	Cesare	Darapti	Bamalip
Celarent	Camestres	Disamis	Calemes
Darii	Festino	Datisi	Dimatis
Ferio	Baroco	Felapton	Fesapo
		Bocardo	Fresison
		Ferison	

Examples

1. Name this syllogism:

No lazy people pass exams.
Some students pass exams.
Therefore,
Some students are not lazy.

2. Write syllogisms:

Write a *Baroco* and a *Datisi* syllogism.

3. Compare:

(How) are *Disamis* and *Dimatis* syllogisms different?

4. Is there more?

Can you formulate a *Pelican* syllogism?

Recall from last time...

Categorical propositions come in 4 different *types*:

	Quantifier	Subject	Copula	Predicate	Type
a	All	S	are	P	Universal affirmative
e	No	S	are	P	Universal negative
i	Some	S	are	P	Particular affirmative
o	Some	S	are not	P	Particular negative

and syllogisms (in standard form) come in 4 different *figures*:

	Figure 1	Figure 2	Figure 3	Figure 4
Major Premise	M - P	P - M	M - P	P - M
Minor Premise	S - M	S - M	M - S	M - S
Conclusion	S - P	S - P	S - P	S - P

As a result, it is possible to classify them, i.e. make an exhaustive list.

Classification of syllogisms

Figure 1	**Figure 2**	**Figure 3**	**Figure 4**
Barbara	Cesare	Darapti	Bamalip
Celarent	Camestres	Disamis	Calemes
Darii	Festino	Datisi	Dimatis
Ferio	Baroco	Felapton	Fesapo
		Bocardo	Fresison
		Ferison	

Example of *Baroco* syllogism (E. Morris)

All fish are orange.

Some shrimp are not orange.

Therefore,

Some shrimp are not fish.

About the empty boxes

Figure 1	Figure 2	Figure 3	Figure 4
Barbara	Cesare	Darapti	Bamalip
Celarent	Camestres	Disamis	Calemes
Darii	Festino	Datisi	Dimatis
Ferio	Baroco	Felapton	Fesapo
		Bocardo	Fresison
		Ferison	

- With 4 types of syllogisms and 4 figures, how many types of syllogisms could we have expected?
- Why isn't there such a thing as a *Bibere* syllogism?

More on orange fish

Let us give this example a little more thought:

All fish are orange.
Some shrimp are not orange.
Therefore,
Some shrimp are not fish.

It is a proper syllogism (it has a name). Let us examine its *content*.

- The major premise is false.
- The minor premise is true.
- The conclusion is true.

- Did we derive something true from something false by following the rules of logic?
- *What is promised to us if we follow the rules of logic?*

Truth values

Recall that we only consider propositions that can be meaningfully said true or false. Given some propositions that are arbitrarily decided to hold true (axioms), the rules of logic allow to combine them to produce new true statements.

The rules indicate how the truth value changes when propositions are combined.

Example:

Negation

The *negation* of a proposition is true if and only if the proposition is false.

$\neg A$ is true exactly when A is false.

Truth tables

This can be expressed in the following table:

A	$\neg A$
T	F
F	T

Another example:

Disjunction

The *disjunction* of two propositions is true if and only if at least one of them is true.

$A \vee B$ is true exactly when A *or* B is (or both are) true.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Truth tables

Conjunction

The *conjunction* of two propositions is true if and only if both of them are.

$A \wedge B$ is true exactly when A *and* B are true.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Truth tables

Exercise: write truth tables for $\neg(A \vee B)$ and $(\neg A) \wedge (\neg B)$

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg(A \vee B)$	$(\neg A) \wedge (\neg B)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

The truth tables are the same... *Could we have predicted that?*

Propositions with identical truth tables are said *logically equivalent*.

Exercises

- Compare the truth tables of $\neg(A \wedge B)$ and $(\neg A) \vee (\neg B)$.
- Write the truth table for $(\neg A) \vee B$.
- $(\neg A) \vee B$ is also written as "$A \Rightarrow B$". Does that make sense?

Implication

If A and B are two propositions, we may form the *implication*:

$$A \Rightarrow B$$

defined as:

$$(\neg A) \vee B$$

with truth table:

A	B	$\neg A$	$A \Rightarrow B$
T	T	F	T
F	T	T	T
T	F	F	F
F	F	T	T

A is called a sufficient condition for B.

B is called a necessary condition for A.

Converses and contrapositives

A	B	$\neg A$	$\neg B$	$A \Rightarrow B$	$B \Rightarrow A$	$(\neg B) \Rightarrow (\neg A)$
T	T	F	F	T	T	T
F	T	T	F	T	F	T
T	F	F	T	F	T	F
F	F	T	T	T	T	T

$B \Rightarrow A$ is called the *converse* of $A \Rightarrow B$.

It is not logically equivalent to $A \Rightarrow B$.

$(\neg B) \Rightarrow (\neg A)$ is called the *contrapositive* of $A \Rightarrow B$.

It is logically equivalent to $A \Rightarrow B$.

This gives a powerful tool to prove things in Mathematics (and elsewhere).

Equivalence

Let us define the *equivalence* of two propositions

$$A \Leftrightarrow B$$

by

$$(A \Rightarrow B) \wedge (B \Rightarrow A).$$

The truth table gives:

A	B	$A \Rightarrow B$	$B \Leftarrow A$	$A \Leftrightarrow B$
T	T	T	T	T
F	T	T	F	F
T	F	F	T	F
F	F	T	T	T

This formalizes the fact that A and B are *logically equivalent* if they have the same truth values.

Modus Ponens

Consider the combination $\mathrm{MP}(A, B)$ defined by:

$$\Big[A \ \wedge \ (A \Rightarrow B)\Big] \Rightarrow B,$$

that is,

$$\neg\Big[A \ \wedge \ (A \Rightarrow B)\Big] \vee B$$

A	B	$A \Rightarrow B$	$A \ \wedge \ (A \Rightarrow B)$	$\neg\Big[A \ \wedge \ (A \Rightarrow B)\Big]$	$\mathrm{MP}(A, B)$
T	T	T	T	F	T
F	T	T	F	T	T
T	F	F	F	T	T
F	F	T	F	T	T

This is true all the time! This is one way syllogisms work...