Zeno's Paradoxes

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The intellectual heritage bequeathed to us by the ancient Greeks was rich indeed. The science of geometry and the entire course of Western philosophy both had their beginnings with Thales. Both enjoyed fantastic development at the hands of his early successors, achieving a surprising degree of perfection during antiquity. During the same period, Aristotle provided the first systematic development of formal logic. But the fertile soil from which all of this grew also gave rise to a series of puzzles which have challenged successive generations of philosophers and scientists right down to the present. These are the famous paradoxes of Zeno of Elea who flourished about 500 B.C.

Zeno was a devoted disciple of the philosopher Parmenides, who had held that reality consisted of one undifferentiated, unchanging motionless whole which was devoid of any parts. Motion, change, and plurality were, according to him, mere illusions. Not too many philosophers could accept this view, and Parmenides was apparently the object of some ridicule from those who disagreed. Zeno's main purpose, it is reported, was to refute those who made fun of his master. His aim was to show that those who believed in motion, change, and plurality were involved in even greater absurdities. Out of perhaps forty such puzzles that he propounded, fewer than ten have come down to us, but they involve some very subtle difficulties. Since motion involves the occupation of different places at different times, these paradoxes strike at the heart of our concepts of space and time.

Bertrand Russell once remarked that "Zeno's arguments, in some form, have afforded grounds for almost all theories of space and time and infinity which have been constructed from his time to our own." This statement was made in 1914, in an essay which contains a penetrating analysis of the paradoxes, but as we shall see, there were problems inherent in these puzzles that escaped even Russell. Such difficulties, in fact, have a direct bearing upon our foregoing discussions of space and geometry, revealing deep problems that we have barely mentioned. [...]

The paradoxes of motion

Our knowledge of the paradoxes of motion comes from Aristotle who, in the course of his discussions, offers a paraphrase of each. Zeno's original formulations have not survived.

Achilles and the Tortoise

Imagine that Achilles, the fleetest of Greek warriors, is to run a footrace against a tortoise. It is only fair to give the tortoise a head start. Under these circumstances, Zeno argues, Achilles can never catch up with the tortoise, no matter how fast he runs. In order to overtake the tortoise, Achilles must run from his starting point A to the tortoise's original starting point T_0 (see Figure 1). While he is doing that, the tortoise will have moved ahead to T_1. Now Achilles must reach the point T_1. While Achilles is covering this new distance, the tortoise moves still farther to T_2.

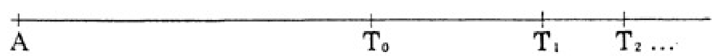

Figure 1

Again, Achilles must reach this new position of the tortoise. And so it continues; whenever Achilles arrives at a point where the tortoise was, the tortoise has already moved a bit ahead. Achilles can narrow the gap, but he can never actually catch up with him. This is the most famous of all of Zeno's paradoxes. It is sometimes known simply as "The Achilles."

The Dichotomy

This paradox comes in two forms, progressive and regressive. According to the first, Achilles cannot get to the end of any racecourse, tortoise or no tortoise; indeed, he cannot even reach the original starting point T_0 of the tortoise in the previous paradox. Zeno argues as follows. Before the runner can cover the whole distance he must cover the first half of it (see Figure 2).

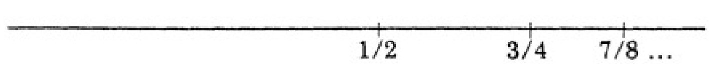

Figure 2

Then he must cover the first half of the remaining distance, and so on. In other words, he must first run one-half, then an additional one-fourth, then an additional one-eighth, etc., always remaining somewhere short of his goal. Hence, Zeno concludes, he can never reach it. This is the progressive form of the paradox, and it has very nearly the same force as Achilles and the Tortoise, the only difference being that in the Dichotomy the goal is stationary, while in Achilles and the Tortoise it moves, but at a speed much less than that of Achilles.

The regressive form of the Dichotomy attempts to show, worse yet, that the runner cannot even get started. Before he can complete the full distance, he must run half of it (see Figure 3). But before he can complete the first half, he must run half of that, namely, the first quarter.

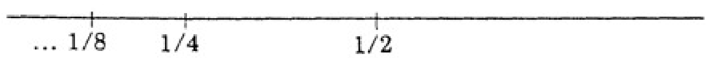

Figure 3

Before he can complete the first quarter, he must run the first eighth. And so on. In order to cover any distance no matter how short, Zeno concludes, the runner must already have completed an infinite number of runs. Since the sequence of runs he must already have completed has the form of a regression, ... $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$ which has no first member, and hence, the runner cannot even get started.

The Arrow

In this paradox, Zeno argues that an arrow in flight is always at rest. At any given instant, he claims, the arrow is where it is, occupying a portion of space equal to itself. During the instant it cannot move, for that would require the instant to have parts, and an instant is by definition a minimal and indivisible element of time. If the arrow did move during the instant it would have to be in one place at one part of the instant, and in a different place at another part of the instant. Moreover, for the arrow to move during the instant would require that during the instant it must occupy a space larger than itself, for otherwise it has no room to move. As Russell says, "It is never moving, but in some miraculous way the change of position has to occur between the instants, that is to say, not at any time whatever." This paradox is more difficult to understand than Achilles and the Tortoise or either form of the Dichotomy, but another remark by Russell is apt: "The more the difficulty is meditated, the more real it becomes."

The Stadium

Consider three rows of objects A, B, and C, arranged as in the first position of Figure 4. Then, while row A remains at rest, imagine rows B and C moving in opposite directions until all three rows are lined up as shown in the second position. In the process, C_1 passes twice as many B's as A's; it lines up with the first A to its left, but with the second B to its left. According to Aristotle, Zeno concluded that "double the time is equal to half."

First Position

	A_1	A_2	A_3	
B_1	B_2	B_3		
		C_1	C_2	C_3

Second Position

A_1	A_2	A_3
B_1	B_2	B_3
C_1	C_2	C_3

Some such conclusion would be warranted if we assume that the time it takes for a C to pass to the next B is the same as the time it takes to pass to the next A, but this assumption seems patently false. It appears that Zeno had no appreciation of relative speed, assuming that the speed of C relative to B is the same as the speed of C relative to A. If that were the only foundation for the paradox we would have no reason to be interested in it, except perhaps as a historical curiosity. It turns out, however, that there is an interpretation of this paradox which gives it serious import.

Suppose, as people occasionally do, that space and time are atomistic in character, being composed of space-atoms and time-atoms of non-zero size, rather than being composed of points and instants whose size is zero. Under these circumstances, motion would consist in taking up different discrete locations at different discrete instants. Now, if we suppose that the A's are not moving, but the B's move to the right at the rate of one place per instant while the C's move to the left at the same speed, some of the C's get past some of the B's without ever passing them. C_1 begins at the right of B_2 and it ends up at the left of B_2, but there is no instance at which it lines up with B_2; consequently, there is no time at which they pass each other – it never happens.

Remarks

It is extremely tempting to suppose, at first glance, that the first three of these paradoxes at least arise from understandable confusions on Zeno's part about concepts of the infinitesimal calculus. It was in this spirit that the American philosopher C.S. Peirce, writing early in the twentieth century, said of Achilles that "this ridiculous little catch presents no difficulty at all to a mind adequately trained in mathematics and logic." There is no reason to think he regarded any of Zeno's other paradoxes more highly.

It has been suggested that Zeno's arguments fit into an overall pattern. Achilles and the Tortoise and the Dichotomy are designed to refute the doctrine that space and time are continuous, while the Arrow and the Stadium are intended to refute the view that space and time have an atomic structure. The paradox of plurality, which will be discussed later, also fits into the total schema. Thus, it has been argued, Zeno tries to cut off all possible avenues to escape from the conclusion that space, time, and motion are not real but illusory.