A Light Dance on the Dust of the Ages

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The ancient Greeks divided the field of mathematics into two main divisions: arithmetic and geometry. Arithmetic was the theory of numbers and geometry the theory of space and its parts. By "numbers" (*arithmoi*), the Greeks meant only the positive integers, what we call the counting numbers. Numbers could be added and multiplied, and a smaller number could be subtracted from a larger. Zero was not a number, nor was there any notion of a negative number. One number could not always be divided by another, since there were no "fractional" numbers. The arithmetic unit, the one, was considered completely indivisible and partless. As Socrates says of arithmetic:

> *It leads the soul forcibly upward, and compels it to discuss the numbers themselves, never permitting anyone to propose for discussion numbers attached to visible or tangible bodies. You know what those who are clever in these matters are like. If, in the course of the argument, someone tries to divide the one itself, they laugh and won't permit it. If you divide it, they multiply it, taking care that one thing never be found to be many parts rather than one.*[1]

The subject matter of geometry included points, (straight) lines, curves, plane figures, solids, etc. Lines, plane figures and solids were instances of magnitudes. In contrast to numbers, magnitudes were taken to be infinitely divisible: a magnitude could always be divided in half, or into any number of equal or unequal parts. The prime examples of magnitudes were straight lines. Lines could be added to one another, and a shorter subtracted from a longer, but the only operation analogous to multiplication does not yield another line: it yields the rectangle that has the two lines as sides. Since space is three-dimensional, three lines can be "multiplied" to form a solid (a rectangular prism), but no similar construction would correspond to "multiplying" four lines. This contrasts with the multiplication of numbers, since the product of two numbers is another number of exactly the same kind.

Magnitudes and numbers, then, had rather little in common for the Greeks. With respect to divisibility, one might even contend that they were fundamentally

[1] *Republic*, p.525e, translation by G. M. A. Grube, revised by C. D. C. Reeve (Plato 1992)

opposed in their natures. Since numbers and magnitudes could both be added, and the smaller subtracted from the larger, certain principles applied to both fields. That is why the axioms of Euclid's Elements include propositions such as "equals added to equals are equals": the axioms were principles that governed both geometry and arithmetic, while the postulates were properly geometrical.

A more interesting commonality between arithmetic and geometry is provided by Eudoxus' theory of proportion. Numbers stand in ratios to one another, and magnitudes of the same kind (such as straight lines) stand in ratios to one another, and a pair of numbers can stand in exactly the same ratio to one another as a pair of lines do. The theory of proportions is presented in Book V of the Elements. Since both numbers and magnitudes can stand in ratios, we begin to see how one might naturally use numbers to represent magnitudes (or magnitudes to represent numbers). We might, for example, be able associate numbers to lines in such a way that the lines stand in exactly the same ratio to one another as their associated numbers do. If there is one way to do this, there are many (for example, doubling all the numbers leaves their ratios unchanged, so the doubled numbers would do as well as the originals), which gives rise to what will much later be called a gauge freedom.

But for the Greeks, numbers would be of only very limited utility as representatives of magnitudes. For, as the Pythagoreans had discovered, magnitudes can stand in ratios that no pair of numbers stand to one another. The famous example is the ratio of the diagonal of a square to one of its sides: no two integers display exactly this proportion. Such pairs of magnitudes were called "incommensurable", since no number of copies of the one, laid end to end, would exactly equal any number of copies of the other.

It is often reported that the Pythagoreans discovered "irrational numbers", or that √2 is irrational, but this is an anachronism. They never recognized what we call rational numbers, much less irrational numbers, and would not have understood "irrational" as an adjective applicable to any individual mathematical object. A magnitude, e.g. the diagonal of a square, is neither "rational" nor "irrational" in itself: it is either commensurable or incommensurable with another magnitude. The fact that the side and diagonal of a square are incommensurable cannot be attributed to anything peculiar about either the side or the diagonal taken individually. Both the side and the diagonal are commensurable with some other magnitudes and incommensurable with others.

What the discovery of incommensurable magnitudes showed was that the structure of ratios among magnitudes is intrinsically richer than the structure of ratios among numbers. That is, the field of geometry presents an inherently more extensive mathematical universe than does the field of arithmetic, as the Greeks understood it. Perhaps this realization lies behind the legend that the Pythagoreans, being on a ship at sea when one of their circle first proved the existence of incommensurable magnitudes, threw the hapless discoverer overboard. The Pythagoreans famously wished to reduce the fundamental essence of all things to number, but geometric investigation demonstrated this to be impossible.