The Mackey Bijection for Reductive Groups and Continuous Fields of Reduced Group C^*-algebras

Angel Román
Joint Work with Nigel Higson

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- The Lie group K, whose Lie algebra is $\mathfrak{k}$ is a *maximal compact subgroup*.
- We can construct an *Iwasawa decomposition* to be seen later.

Cartan Motion Group

- The *Cartan motion group* of G is defined as $G_0 = K \ltimes (\mathfrak{g}/\mathfrak{k})$, with composition $(k, X) \cdot (k_0, X_0) = (kk_0, \mathrm{Ad}_{k_0^{-1}}(X) + X_0)$.

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- Example: $G = SL(2, \mathbb{R})$ is a real reductive group. Its Cartan motion group is $G_0 = SO(2) \ltimes \mathfrak{sl}(2, \mathbb{R})/\mathfrak{so}(2)$.
- Here $\mathfrak{sl}(2, \mathbb{R})$ are the traceless matrices and matrices in $\mathfrak{so}(2)$ has the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

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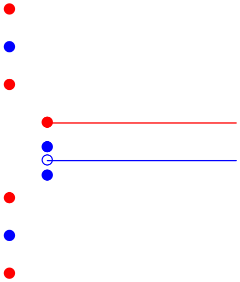
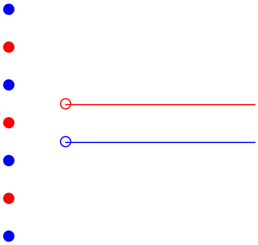
The Mackey Bijection

There is a one-to-one correspondence between $\hat{G}_0$ and $\hat{G}_r$.

Example

Irreducible Tempered Dual of $SL(2,\mathbb{R})$ $\cong$ Unitary Dual of $SO(2) \ltimes \left(\mathfrak{sl}(2,\mathbb{R})/\mathfrak{so}(2)\right)$





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- Let $\sigma \in \hat{M}$ with representation space H_σ and let $\nu \in \mathfrak{a}^*$, a real linear functional on $\mathfrak{a}$.
- A *principal series representation* of G is a unitary induced representation

$$\pi_{\sigma,\nu} = \mathrm{Ind}_P^G\big(\sigma \otimes \exp(i\nu) \otimes 1\big).$$

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- Two principal series $\pi_{\sigma,\nu}$ and π_{σ_0,ν_0} are unitarily equivalent if and only if there is a $w \in N_K(\mathfrak{a})$ such that $\sigma \cong w\sigma$ and $\nu_0 = w\nu$.

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$$\hat{G}_0 \cong \{(\tau_\chi, \chi) | \chi \in \hat{\mathfrak{p}}, \tau_\chi \in \hat{K}_\chi\}/K.$$

Completing the Mackey Bijection

- Let $\mathfrak{a}$ be maximal abelian Lie algebra in $\mathfrak{p}$. Let $\mathfrak{a}^{\perp}$. Every $\chi \in \mathfrak{p}$ is conjugate by some $k \in K$ to $\chi_0 \in \mathfrak{p}$ such that χ_0 vanishes on $\mathfrak{a}^{\perp}$.

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Mackey Bijection for Complex Reductive Lie Group

$$\hat{G}_0 \cong (\hat{\mathfrak{a}} \times \hat{M})/W \cong (\hat{A} \times \hat{M})/W \cong \hat{G}_r.$$

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- There is a unique smooth manifold structure such that the mapping $\mathbb{N}_G K \to G \times \mathbb{R}$ defined by

$$\begin{aligned} (k \exp(X), t) &\mapsto (k \exp(t^{-1}X), t) && \text{if } t \neq 0 \\ (k, X, 0) &\mapsto (k \exp(X), 0) && \text{if } t = 0 \end{aligned}$$

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  is a diffeomorphism.
- The normal bundle has a group structure.

Definition

We call the set $\mathbb{N}_K G$ together with its smooth structure and group structure, the *deformation space*.

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- Let $\{C_r^*(G_t)\}$ be a family of C^*-algebras associated to the deformation space.
- Let $f \in C_c^\infty(\mathbb{N}_G K)$. Define $f_t : G_t \to \mathbb{C}$ by
  $f_t(g) = f(g, t)$ if $t \neq 0$ and
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The sections $\{f_t\}$ defined from $f \in C_c^\infty(\mathbb{N}_G K)$ generates the *continuous sections* of $\{C_r^*(G_t)\}$, making it a *continuous field of C^*-algebras associated to the deformation space.*

Main Result for Complex Reductive Groups

Let $\{f_t\}$ be a continuous section of the continuous field of C^*-algebras associated to the deformation space $\{C_r^*(G_t)\}$.

Theorem

There is a one-parameter group of rescaling automorphisms $\{\alpha_t\}_{t\neq 0}$ such that

$$\lim_{t\to 0} \alpha_t(f_t)$$

exists and defines an embedding of C^*-algebras

$$\alpha : C_r^*(G_0) \hookrightarrow C_r^*(G).$$

Fourier Structure Theorem

Recall

$$\hat{G}_r \cong (\hat{M} \times \hat{A})/W,$$

where $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Define

$$\pi_\sigma : C_r^*(G) \to C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^\sigma))$$

by $\pi_\sigma(f)(\nu) = \pi_{\sigma,\nu}(f)$, where $\pi_{\sigma,\nu} \in \hat{G}_r$. Here, $\sigma \in \hat{M}$ and $\nu \in \hat{A}$.

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Theorem

The principal series representations induce a C^*-algebra isomorphism

$$\pi = \bigoplus_{\sigma \in \hat{M}} \pi_\sigma : C_r^*(G) \xrightarrow{\cong} \left[\bigoplus_{\sigma \in \widehat{M}} C_0\left(\mathfrak{a}^*, \mathcal{K}(L^2(K)^\sigma)\right) \right]^W .$$

Intertwining Operators

There are (normalized) *intertwining operators*

$$\mathcal{A}(w, \sigma, \nu) : \operatorname{Ind}_P^G H_\sigma \to \operatorname{Ind}_P^G H_{w\sigma}$$

such that $\pi_{w\sigma, w\nu}\mathcal{A}(w, \sigma, \nu) = \mathcal{A}(w, \sigma, \nu)\pi_{\sigma,\nu}$.

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Now for each $w \in W_\sigma$, there are (normalized) intertwining operators $\mathcal{A}(w,\sigma,\nu)$ such that $\mathcal{A}(w,\sigma,\nu)\pi_\sigma(f)(\nu) = \pi_\sigma(f)(w\nu)\mathcal{A}(w,\sigma,\nu)$.

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$$C^*_r(G) \cong \bigoplus_{\sigma \in \hat{M}/W} C_0(\mathfrak{a}^*_{\sigma,+}, \mathcal{K}(L^2(K)^\sigma)).$$

- Define, for $t > 0$ and $\sigma \in \hat{M}/W$, an automorphsim

$$\alpha_{\sigma,t} : C_0\big(\mathfrak{a}^*_{\sigma,+}, \mathcal{K}(L^2(K)^\sigma)\big) \longrightarrow C_0\big(\mathfrak{a}^*_{\sigma,+}, \mathcal{K}(L^2(K)^\sigma)\big)$$

by $\alpha_{\sigma,t}(f)(\nu) = f(t^{-1}\nu)$.

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  by $\alpha_{\sigma,t}(f)(\nu) = f(t^{-1}\nu)$.
- Define the automorphisms $\alpha_t : C^*_r(G) \to C^*_r(G)$ by

$$\alpha_t = \pi^{-1} \circ \Big(\bigoplus \alpha_{\sigma,t}\Big) \circ \pi.$$

Theorem about Limit

Define $\lambda_t : C_r^*(G_t) \xrightarrow{\cong} C_r^*(G)$ by the formula

$$f_t \longmapsto \left[g \mapsto |t|^{-d} f_t(g) \right],$$

for $f_t \in C_c^\infty(G_t)$.

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Theorem

If $\{f_t\}$ is a continuous section of the continuous field $\{C_r^*(G_t)\}$, then the limit

$$\lim_{t \to 0} \alpha_t(\lambda_t(f_t))$$

exists in $C_r^*(G)$.

Embedding Morphism

Let $f \in C_r^*(G_0)$. Extend f in any way to a continuous section $\{f_t\}$ of $\{C_r^*(G_t)\}$ and then form the limit

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in $C_r^*(G)$.

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in $C_r^*(G)$.

Theorem

The above formula defines an embedding of C^*-algebras

$$\alpha \colon C_r^*(G_0) \longrightarrow C_r^*(G).$$

Characterization of Mackey Bijection

Theorem

There is a unique bijection

$$\mu : \hat{G}_r \to \hat{G}_0$$

such that for every $\pi \in \hat{G}_r$, the element $\mu(\pi) \in \hat{G}_0$ may be realized as a unitary subrepresentation of $\pi \circ \alpha$.

- This is given by the Mackey bijection

$$\pi_{\sigma,\nu} \mapsto \phi_{\nu,\tau_\nu},$$

already established.

Considering Real Reductive Groups

From my 2019 dissertation:

Let $G = SL(2, \mathbb{R})$. The tempered representations are:

- the principal series $\pi_{\epsilon,\nu}$ and $\pi_{1,\nu}$ induced from the minimal parabolic subgroup $P = MAN$. We can parametrize ν by $\mathbb{R}$.
- the discrete series D_n, where n is an integer and $|n| \geq 2$. Then

$$C_r^*(G) \cong \bigoplus \mathcal{K}(H_n) \oplus C_0(\mathbb{R}, \mathcal{K}(\operatorname{Ind}_P^G H_\epsilon))^{\mathbb{Z}/2\mathbb{Z}} \oplus C_0(\mathbb{R}, \mathcal{K}(\operatorname{Ind}_P^G H_1))^{\mathbb{Z}/2\mathbb{Z}}.$$

- On the discrete series component, define $\alpha_{n,t} : \mathcal{K}(H_n) \to \mathcal{K}(H_n)$ by $\alpha_{n,t}(T) = T$ for $T \in \mathcal{K}(H_n)$.

- The component $C_0(\mathbb{R}, \mathcal{K}(\operatorname{Ind}_P^G H_1))^{\mathbb{Z}/2\mathbb{Z}}$ is equivalent to $C_0([0,\infty), \mathcal{K}(\operatorname{Ind}_P^G H_1))$.
- define $\alpha_{1,t}$ in the usual way: $\alpha_{1,t}(F(t^{-1}\nu))$.
- The component $C_0(\mathbb{R}, \mathcal{K}(\operatorname{Ind}_P^G H_\epsilon))^{\mathbb{Z}/2\mathbb{Z}}$ is equivalent to $C_0(\mathbb{R}, \mathcal{K}(\operatorname{Ind}_P^G H_\epsilon))$ *with* the condition that $\mathcal{A}(w,\epsilon,0)F(0) = F(0)\mathcal{A}(w,\epsilon,0)$.
- But $\mathcal{A}(w,\epsilon,0)$ is not a scalar multiple of the identity map.
- This correspond to the fact that $\pi_{\epsilon,0}$ is reducible.
- We can still define $\alpha_{\epsilon,t}$ in the usual way since $t > 0$.
- Using these rescaling map, we can induce an embedding of C^*-algebras

$$\alpha : C_r^*(G_0) \hookrightarrow C_r^*(G).$$

Ongoing Work: Real Reductive Groups

joint work with Nigel Higson and Pierre Clare

Let $G = SL(n, \mathbb{R})$ where $n = 4$.

- There are three classes of *cuspidal* parabolic subalgebras of $\mathfrak{g}$

$$P_0 = M_0 A_0 N_0 \qquad P_1 = M_1 A_1 N_1 \qquad P_2 = M_2 A_2 N_2.$$

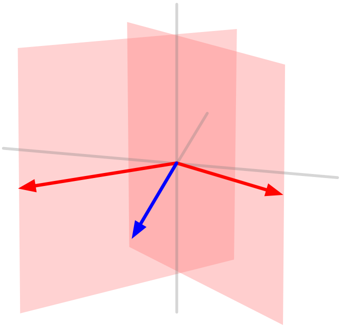
- Here $M_0 = \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) : \epsilon = \pm 1 \text{ and } \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1\}$.
- Here elements of M_1 has the block-diagonal form $SL_{\pm}(2, \mathbb{R}) \times \mathrm{diag}(\epsilon_1, \epsilon_2)$ where $\epsilon = \pm 1$ and the determinant is 1.
- Here elements of M_2 has the block-diagonal form $SL_{\pm}(2, \mathbb{R}) \times SL_{\pm}(2, \mathbb{R})$.

$$C_r^*(G) \cong \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathcal{K}(\mathrm{Ind}_P^G H_{P,\sigma}))^{W_{P,\sigma}}$$

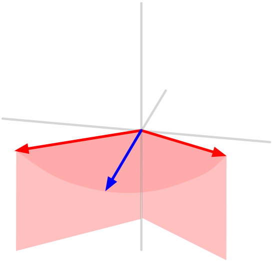
where $[P, \sigma]$ is the equivalent classes, P is a cuspidal parabolic subgroup and σ is a discrete series of M.

Consider only the minimal parabolic subgroup component $P_0 = M_0 A_0 N_0$.

- Goal: we would like to find a fundamental domain $\mathfrak{a}^*_{P,\sigma,+}$ under the action of $W_{P,\sigma}$.
- The finite group $W_{P,\sigma}$ has order 8 and can be generated by 3 elements of order one: w_1, w_2, r. Not a Weyl group.
- $W'_{P,\sigma} = \langle w_1, w_2 \rangle$ is a Weyl group and $R = \langle r \rangle$.



- Let $\mathfrak{a}^*_{P,\sigma,+}$ be the fundamental domain under the action of $A'_{P,\sigma}$.
- Then the component $C_0(\mathfrak{a}^*_P, \mathcal{K}(\operatorname{Ind}_P^G H_{P,\sigma}))^{W_{P,\sigma}}$ becomes $C_0(\mathfrak{a}^*_P, \mathcal{K}(\operatorname{Ind}_P^G H_{P,\sigma}))^R$.
- We would like to find a fundamental domain under R. See figure.
- Problem: there are $\nu \in \mathfrak{a}^*_{P,\sigma,+}$ such that $\mathcal{A}(r,\sigma,\nu)F(\nu) = F(\nu)\mathcal{A}(r,\sigma,\nu)$ and $\mathcal{A}$ is *not* a scalar multiple of the identity. Under the naive scaling automorphism, $\alpha_t(F)$ would not be invariant under R.



Thank You!