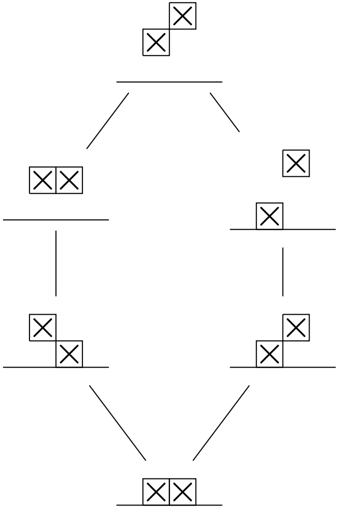


GAG Seminar
April 17, 2024

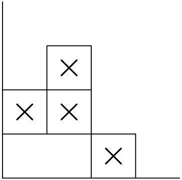
Shellability of Kohnert posets

by
Celia Kerr
and
Nick Russoniello

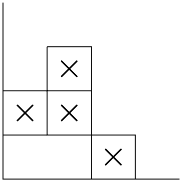
Joint work with
N. Mayers, Ph.D. (NC State)

William & Mary

Diagrams



Kohnert Moves



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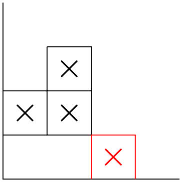

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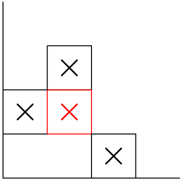
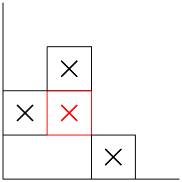
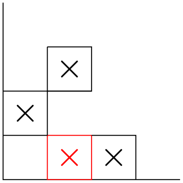


Kohnert Moves





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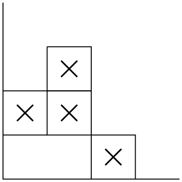

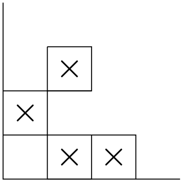

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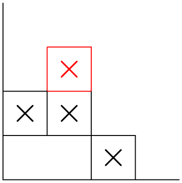

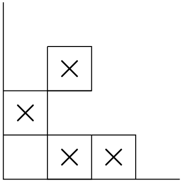

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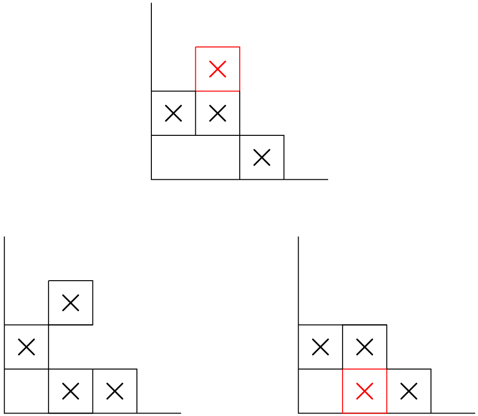

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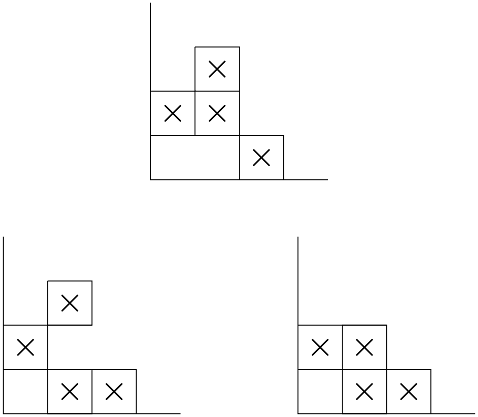

Kohnert Diagrams

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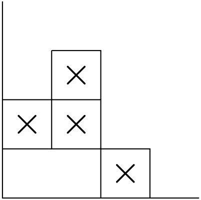
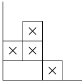


Kohnert Diagrams



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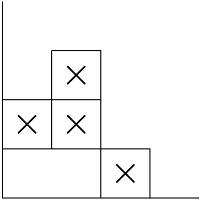

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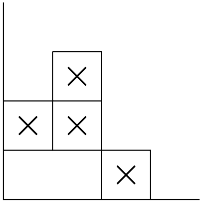

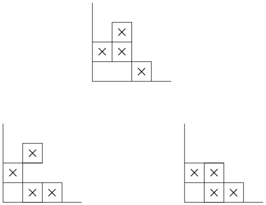

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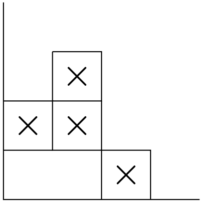

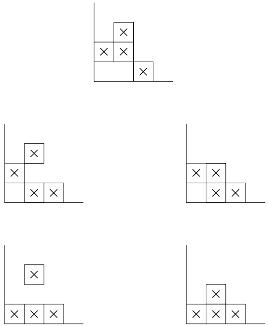

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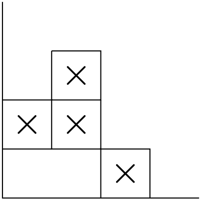
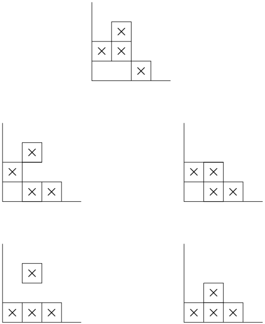
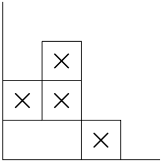


$KD(D)$:



Kohnert Polynomials

D:



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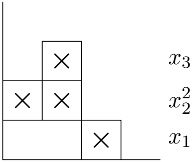
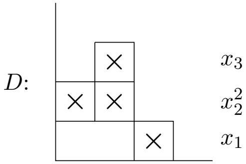


$$wt(D) = x_1 x_2^2 x_3$$

Kohnert Polynomials



$$wt(D) = x_1 x_2^2 x_3$$

$$\mathfrak{K}_D = \sum_{T \in KD(D)} wt(T)$$

Kohnert Polynomials

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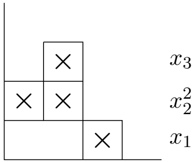

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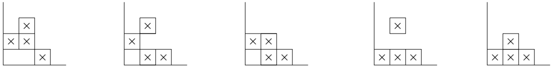
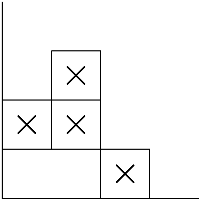


$$\mathfrak{K}_D = \sum_{T \in KD(D)} wt(T) = x_1 x_2^2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2^2 + x_1^3 x_3 + x_1^3 x_2$$

Kohnert Posets

D:



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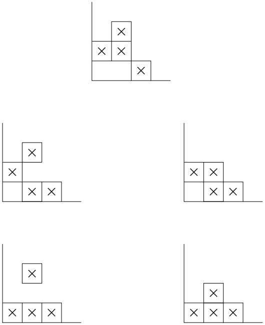
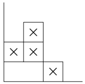


Kohnert Posets

$\underline{\mathcal{P}(D)}$



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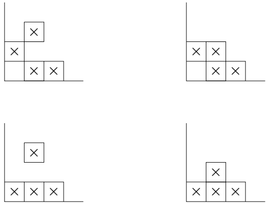

Kohnert Posets

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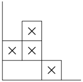
A **poset** is a set together with a relation that is



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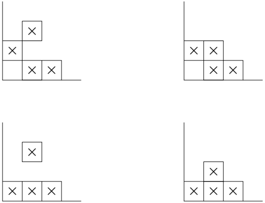


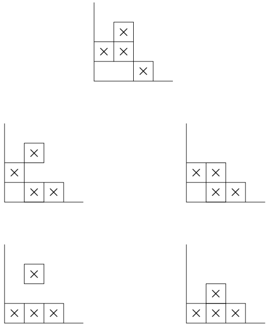
Kohnert Posets

$\mathcal{P}(D)$

A **poset** is a set together with a relation that is

- Reflexive

$KD(D):$



Kohnert Posets

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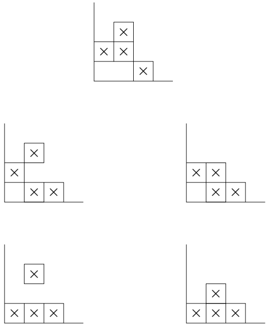
A **poset** is a set together with a relation that is

- Reflexive
- Antisymmetric

$KD(D)$:



Kohnert Posets

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A **poset** is a set together with a relation that is

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- Transitive

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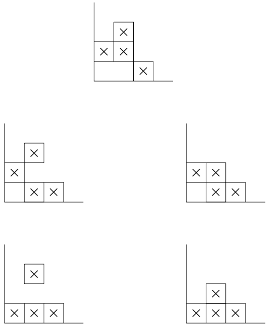

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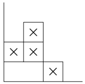
$D_1 \preceq D_2$



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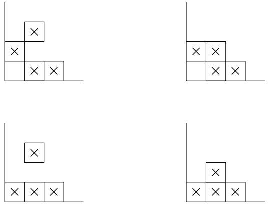

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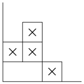

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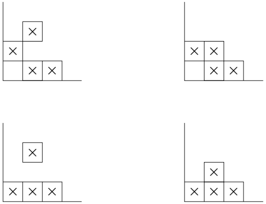

Kohnert Posets

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A **poset** is a set together with a relation that is

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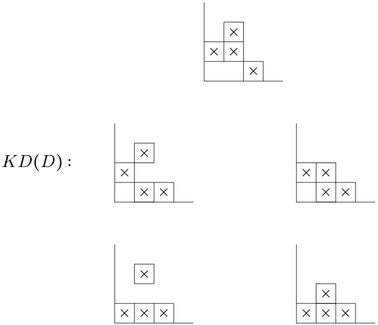
$$D_1 \leq D_2$$

$$\Longleftrightarrow$$

$$D_2 \xrightarrow{\text{Kohnert}} D_1$$



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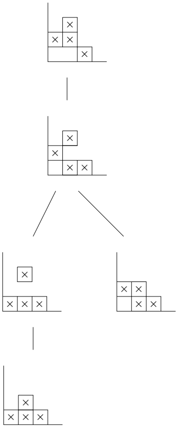

Kohnert Posets

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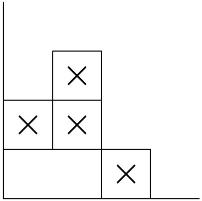

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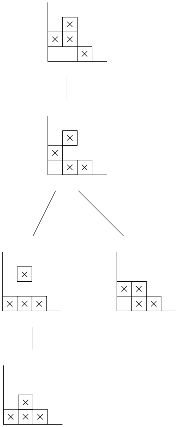

A poset is **bounded** if it has a unique maximal and minimal element.

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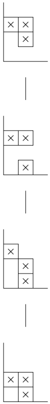

Bounded

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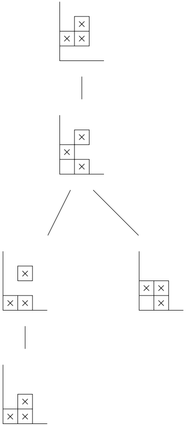

Not Bounded

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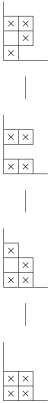

Graded

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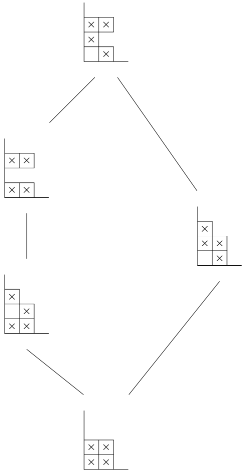

Not Graded

Abstract Simplicial Complexes

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Definition

On a set of vertices V, an **abstract simplicial complex** Δ is a finite collection of subsets of V, called **faces**, satisfying

$$\tau \subseteq \sigma \in \Delta \implies \tau \in \Delta.$$

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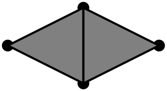

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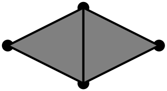
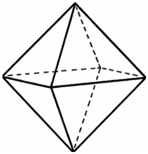
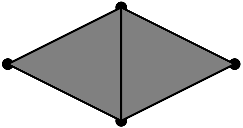
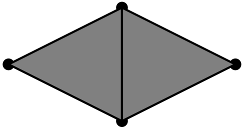
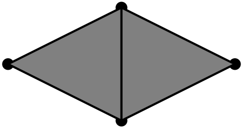




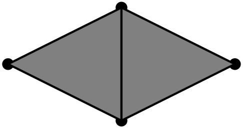




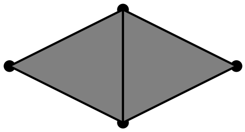
- Vertices - dimension 0



- Vertices - dimension 0
- Segments - dimension 1

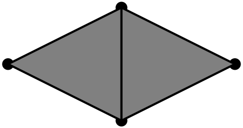


- Vertices - dimension 0
- Segments - dimension 1
- Triangles - dimension 2



- Vertices - dimension 0
- Segments - dimension 1
- Triangles - dimension 2

Faces of maximal dimension are called **facets**.



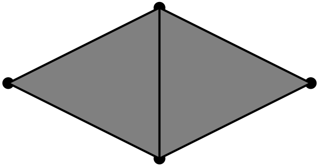
- Vertices - dimension 0
- Segments - dimension 1
- Triangles - dimension 2

Faces of maximal dimension are called **facets**.

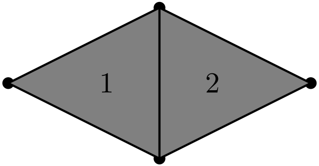
Definition

A simplicial complex Δ is called **shellable** if its facets can be arranged into a total order $F_1, \cdots, F_t$ in such a way that the subcomplex $\left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for $2 \le k \le t$. Such an ordering of facets is called a **shelling**.

Examples



Examples



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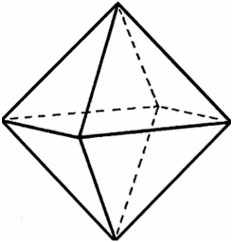

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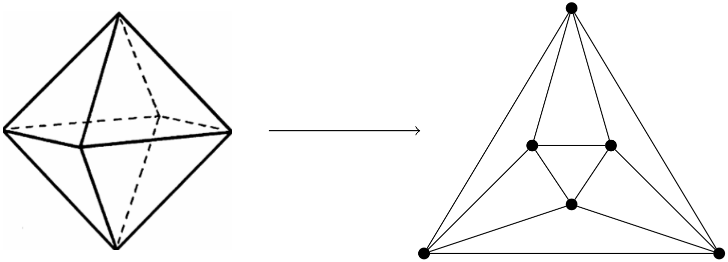

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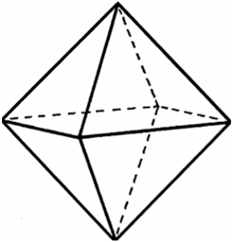

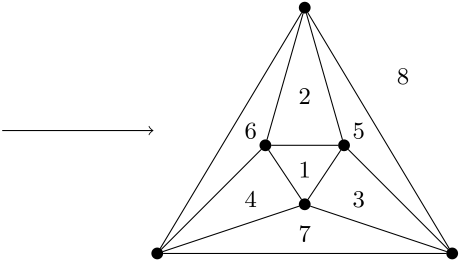

Non-Examples

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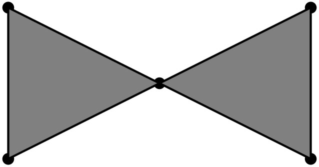

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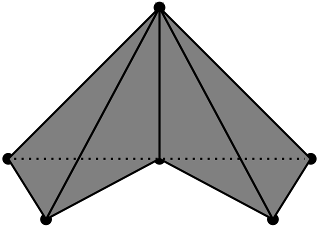

Order Complex of a Poset $(\mathcal{P}, \leq)$

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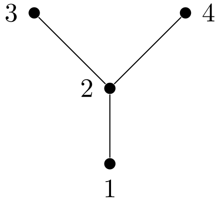

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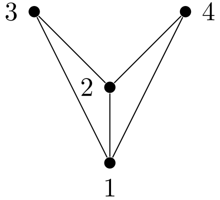

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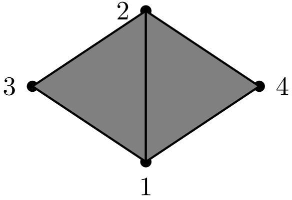
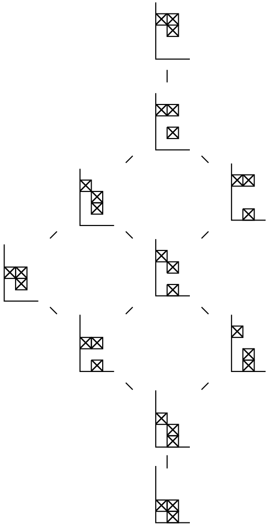


Question: Can we characterize when Kohnert posets are shellable?




You're gonna need a
different notion of shellability

EL-Shellable

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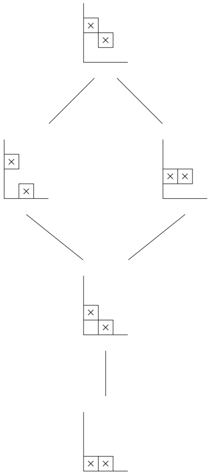

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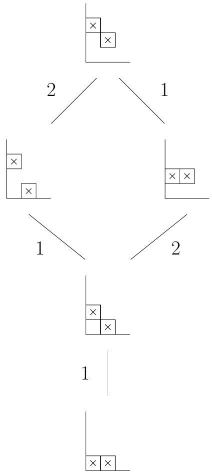

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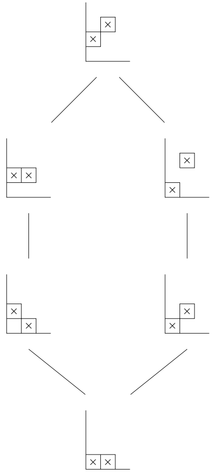

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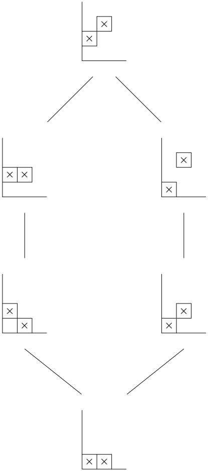

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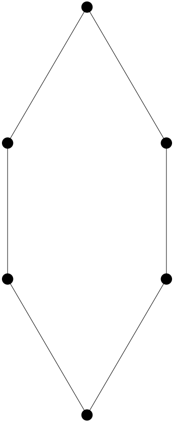

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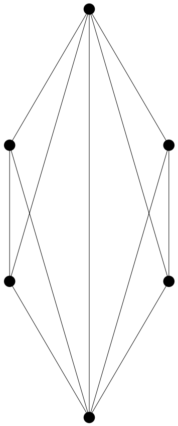

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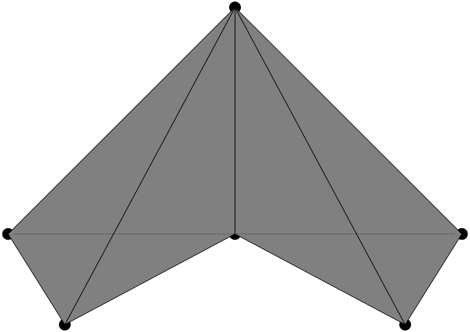
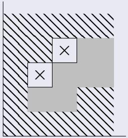
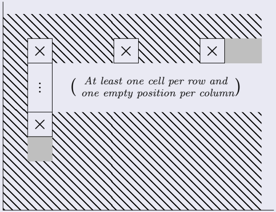


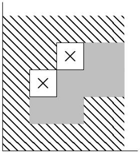
Not Shellable

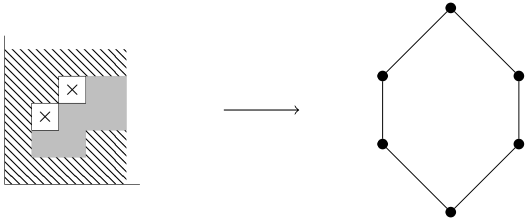
Theorem

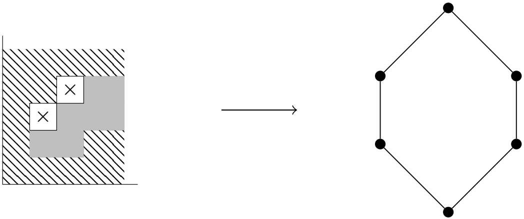
Let D be a diagram. If there exists $D^* \in KD(D)$ such that D^* contains either of the subdiagrams illustrated below, then $\mathcal{P}(D)$ is not (EL-)shellable.

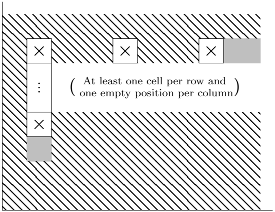











At least one cell per row and
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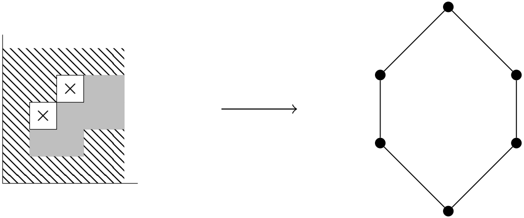

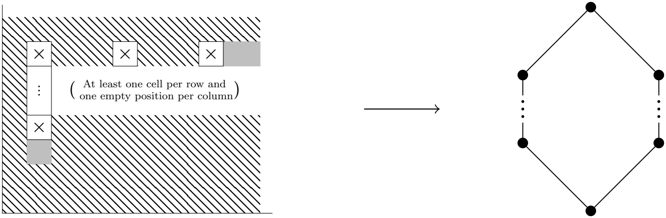
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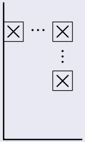
Definition

A **hook diagram** is a diagram which, after removing empty columns, is of the form illustrated below or can be formed from such a diagram by applying Kohnert moves.



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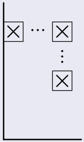
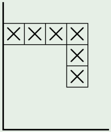
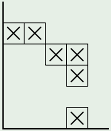


Example





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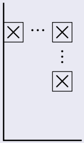
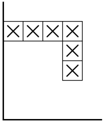
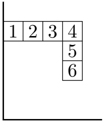
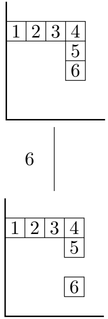


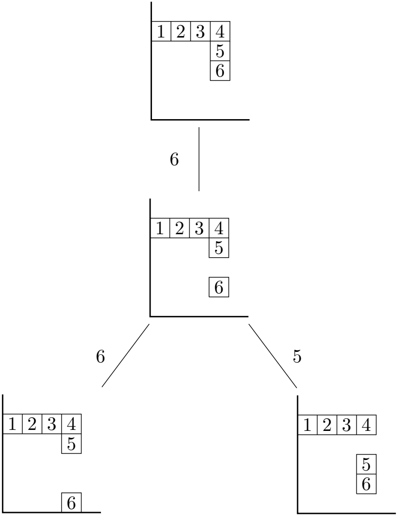
Theorem

If D is a hook diagram, then $\mathcal{P}(D)$ is EL-shellable.




1
2
3
4
5
6




1 2 3 4
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Lemma

Let D be a hook diagram. If $D' = D\Big\downarrow_{(r-k,c)}^{(r,c)}$ *and* $D' \lessdot D$*, then* $k = 1$.

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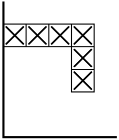

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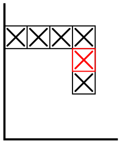

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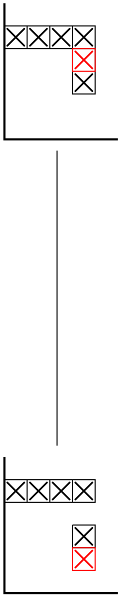

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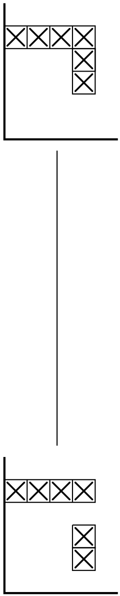

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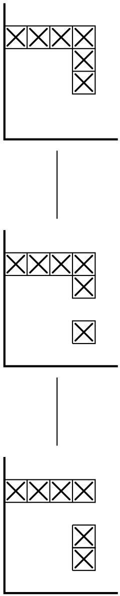
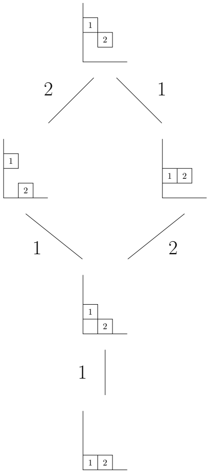


Theorem

Let D be a hook diagram and $I = [D_1, D_2]$ be an interval in $\mathcal{P}(D)$. If $\mathcal{C}_1, \mathcal{C}_2$ are maximal chains in I, then the multiset of labels for $\mathcal{C}_1$ equals the multiset of labels for $\mathcal{C}_2$.



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