Positivity preservers
forbidden to operate on diagonal blocks

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Positive semidefinite matrices

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Suppose $n \geq 1$ is an integer, and $A := (a_{ij})$ and $B := (b_{ij}) \in \mathbb{C}^{n \times n}$ are PSD. Then

$$A \circ B := (a_{ij} b_{ij}) \qquad \text{is PSD.}$$

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In particular, for integer $m \geq 0$ (under the convention $0^0 := 1$)

$$A^{\circ m} := A \circ A \circ \cdots \circ A = \left(a_{ij}^m\right) \text{ is PSD.}$$

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More succinctly, the infinite sums of Herz functions

$$\sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k, \qquad \text{where } c_{m,k} \geq 0 \text{ for all } m, k \geq 0 \tag{1}$$

when operating entrywise on any PSD matrix, preserve its positivity. (Pólya and Szegö (1925) for real PSD matrices.)

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Loewner positive functions

Let $\mathcal{P}$ denote a subset of all Hermitian PSD matrices, and f be a complex valued function such that $f[A] := \big(f(a_{ij})\big)$ is well-defined for all $A := (a_{ij}) \in \mathcal{P}$. Then f is Loewner positive over $\mathcal{P}$ if

$$f[A] = \begin{pmatrix} f(a_{11}) & f(a_{12}) & \cdots & f(a_{1n}) \\ f(a_{21}) & f(a_{22}) & \cdots & f(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_{n1}) & f(a_{n2}) & \cdots & f(a_{nn}) \end{pmatrix} \text{ is PSD for all } A = (a_{ij}) \in \mathcal{P}.$$

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Loewner positive functions can be explored

- For different choices of $\mathcal{P} \begin{cases} = \mathbb{P}_n \ \left(= \cup_{k=1}^{n} \mathbb{P}_k \right) & \text{(fixed-dimensional)} \\ = \cup_{k=1}^{\infty} \mathbb{P}_k & \text{(dimension-free)} \end{cases}$

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- For entrywise variations of "$f[-]$" — main focus in the talk.

Overview

1. Metric geometry, positivity, and Loewner positive functions.
2. First entrywise variant of "$f[-]$" and its generalization.
3. Main results for the *new* variants of "$f[-]$."
4. Proofs.

Recall. . .

Positive semidefinite matrices

An $n \times n$ Hermitian matrix A is positive semidefinite (PSD) if one of the following holds:

1. The quadratic form $z^* A z \geq 0$ for all $z \in \mathbb{C}^n$.
2. All principal minors of A are nonnegative.
3. All eigenvalues of A are nonnegative.

Notations

- $\mathbb{P}_n$ denotes the set of all $n \times n$ PSD matrices.
- $\mathbb{P}_n(I) := \mathbb{P}_n \cap I^{n \times n}$, where $I \subseteq \mathbb{C}$.

PART 1

Metric geometry,
Positivity,
&
Loewner positive functions.

Recall...

Metric spaces

A metric space X is a nonempty set, with a metric $d : X \times X \to \mathbb{R}$ such that

1. *Positivity:* $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$.
2. *Symmetry:* $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Positive definite functions

Suppose (X, d) is metric space, and $f : [0, \infty) \to \mathbb{R}$. f is positive definite over X, if for every set of points $x_1, x_2, \ldots, x_n$, the matrix

$$\big(f \circ d(x_i, x_j)\big)_{i,j=1}^{n} \qquad \text{is positive semidefinite.}$$

Metric embedding into $\mathbb{R}^r$

Metric space embedding into Euclidean spaces $\mathbb{R}^r$, for integer $r \geq 1$.

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Theorem (Schoenberg [*Ann. of Math.* 1935, *Trans. Amer. Math. Soc.* 1938])

Let $(X = \{x_0, x_1, \ldots, x_n\}, d)$ be a metric space. The following are equivalent.

1. The metric space (X, d) embeds isometrically into Euclidean space $\mathbb{R}^r$, for some integer $r \geq 1$.
2. The $n \times n$ (modified) Cayley–Menger matrix

$$A := \left(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2\right)_{i,j=1}^{n} \text{ is PSD.}$$

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3. The $(n+1) \times (n+1)$ matrix

$$B := \left(\exp\left(-\sigma^2 d(x_i, x_j)^2\right)\right)_{i,j=0}^n \text{ is PSD}$$

   along any sequence of nonzero scalars $\sigma \in \mathbb{R}$ converging to zero.

Moreover, the smallest such r is the rank of the matrix A.

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Metric space embedding into unit spheres S^{r-1}, for integer $r \geq 2$, and S^{∞}.

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- Impose *arclength metric* d_a on S^{r-1}, where $r \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, defined as

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- Suppose $\xi : X \hookrightarrow S^{r-1}$ is an isometric embedding, where
  - $X = \{x_1, \ldots, x_n\}$ is with metric $d(\cdot, \cdot) \leq \pi$, and
  - $r \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

Then

$$\cos \circ d(x_i, x_j) = \cos \circ d_a(\xi(x_i), \xi(x_j)) = \langle \xi(x_i), \xi(x_j) \rangle.$$

Metric embedding into unit/Hilbert sphere (cont.)

This means

$$C := \big(\cos \circ d(x_i, x_j)\big)_{i,j=1}^{n} \text{ is a Gram matrix (of rank at most } r\text{).}$$

Therefore C is PSD, and $\cos$ is positive definite over S^{r-1}, for $r \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

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In fact, the converse also holds:

Theorem (Schoenberg [*Ann. of Math.* 1935])

Let $(X = \{x_1, x_2, \ldots, x_n\}, d)$ be a metric space with diameter at most π. The following are equivalent.

1. The metric space (X, d) embeds isometrically into the unit sphere S^{r-1}, but not into S^{r-2}.
2. $C := \big(\cos \circ d(x_i, x_j)\big)_{i,j=1}^{n}$ is PSD of rank r.

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In particular, (X, d) isometrically embeds into S^∞ if and only if

cosine is positive definite over X.

Loewner positive functions over correlation matrices

More examples of such positive definite functions over S^∞?

- Schoenberg classified functions f such that $f \circ \cos$ is positive definite over S^∞.

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Observations

- $C := \big(\cos \circ d_a(x_i, x_j)\big)_{i,j=1}^n$ is a correlation matrix for $x_1, x_2, \ldots, x_n \in S^\infty$.

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- $f \circ \cos$ is positive definite over $S^\infty \iff f$ is Loewner positive over the set of correlation matrices.

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Theorem (Schoenberg [*Duke Math. J.* 1942])

Let $I = [-1, 1]$, and $f : I \to \mathbb{R}$ be continuous. The following are equivalent.

1. $f \circ \cos$ is positive definite over S^∞.
2. $f[A] := \big(f(a_{ij})\big)$ is PSD for all correlation matrices $A = (a_{ij})$.
3. $f(x) = \sum_{k \geq 0} c_k x^k$ for all $x \in I$, where $c_k \geq 0$ for all $k \geq 0$.

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Lemma (Pólya–Szegö 1925)

Suppose $f(x) := \sum_{k\geq 0} c_k x^k$ is a power series with all $c_k \geq 0$, that converges over $I \subseteq \mathbb{R}$. Then $f[A] := \big(f(a_{ij})\big)$ is PSD for all PSD $A := (a_{ij})$, where $a_{ij} \in I$.

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Theorem (Rudin [*Duke Math. J.* 1959],
Christensen–Ressel [*Trans. Amer. Math. Soc.* 1978])

Let $I = (-\rho, \rho)$ for $0 < \rho \leq \infty$, and $f : I \to \mathbb{R}$. The following are equivalent:

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Rudin conjectured the complex version: $\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$, with $c_{m,k} \geq 0$.

Loewner positive functions over all dimensions (cont.)

Theorem (Herz [*Ann. Inst. Fourier (Grenoble)* 1963],
FitzGerald–Micchelli–Pinkus [*Linear Algebra Appl.* 1995])

Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$, and $f : I \to \mathbb{C}$. The following are equivalent:

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Similar results were obtained for doubly nonnegative matrices by Vasudeva:

Theorem (Vasudeva [*Indian J. Pure Appl. Math.* 1979],
Guillot–Khare–Rajaratnam [*Trans. Amer. Math. Soc.* 2017])

Let $I = (0,\rho)$ or $[0,\rho)$ for $0 < \rho \leq \infty$, and $f : I \to \mathbb{R}$. The following are equivalent:

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Loewner positive functions over all dimensions (summary)

(*Dimension-free*) Loewner positive functions over $\mathcal{P} := \cup_{n=1}^{\infty} \mathbb{P}_n(I)$.

- Exactly the class of infinite sums of Herz functions over I, for $I = D(0, \rho)$, $(-\rho, \rho)$, $(0, \rho)$, and $[0, \rho)$.

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Observe that, for all integers $\alpha \geq 0$,

$$f_\alpha(x) := x^\alpha, \ x \geq 0, \text{ is Loewner positive over } \mathbb{P}_n((0, \infty)), \text{ for all } n \geq 1.$$

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Natural to ask for the classification of all $\alpha \in \mathbb{R}$, such that

$$f_\alpha \text{ is Loewner positive over } \mathbb{P}_n((0, \infty)), \text{ for a fixed } n \geq 2.$$

Loewner positive functions over all dimensions (summary)

(*Dimension-free*) Loewner positive functions over $\mathcal{P} := \cup_{n=1}^{\infty} \mathbb{P}_n(I)$.

- Exactly the class of infinite sums of Herz functions over I, for $I = D(0,\rho)$, $(-\rho,\rho)$, $(0,\rho)$, and $[0,\rho)$.
- In particular, these are absolutely monotonic for real domains I. However, such is not the case in the *fixed-dimensional* setting, i.e. when $\mathcal{P} = \mathbb{P}_n(I)$, for fixed $n \geq 1$.

Example 1

Observe that, for all integers $\alpha \geq 0$,

$$f_\alpha(x) := x^\alpha, \ x \geq 0, \ \text{is Loewner positive over } \mathbb{P}_n((0,\infty)), \ \text{for all } n \geq 1.$$

Natural to ask for the classification of all $\alpha \in \mathbb{R}$, such that

$$f_\alpha \text{ is Loewner positive over } \mathbb{P}_n((0,\infty)), \ \text{for a fixed } n \geq 2.$$

Theorem (FitzGerald–Horn [*J. Math. Anal. Appl.* 1977])

Let $n \geq 2$ be an integer, and $\alpha \in \mathbb{R}$. Suppose $f_\alpha(x) := x^\alpha$ for $x \geq 0$. Then,

$$f_\alpha \text{ is Loewner positive over } \mathbb{P}_n((0,\infty)) \iff \alpha \in [n-2,\infty) \cup \mathbb{Z}_{\geq 0}.$$

Loewner positive functions over fixed dimension

Example 2

Polynomials with nonnegative coefficients are all the polynomials that are dimension-free Loewner positive.

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- Feature *non-absolutely monotonic* Loewner positive functions in the fixed-dimensional setting.

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- In the dimension-free setting, however, *non-absolutely monotonic* Loewner positive functions cannot exist. (Schoenberg, Rudin, Herz, Vasudeva.)

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In this talk

- First examples of (real) **non-absolutely monotonic dimension-free** Loewner positive functions — for a refined entrywise variant of "$f[A]$."
- This translates into discovering dimension-free Loewner positive functions over complex PSD matrices which are not an infinite sum of Herz functions.

PART 2

An entrywise variant,

&

Generalizations.

An entrywise variant motivated by modern applications

Consider the following functions for $\epsilon > 0$:

- Hard thresholding: $f_\epsilon^H(x) := \begin{cases} x, & \text{if } |x| > \epsilon, \\ 0, & \text{otherwise.} \end{cases}$
- Soft thresholding: $f_\epsilon^S(x) := \text{sgn}(x)(|x| - \epsilon)_+$.

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In modern high-dimensional probability and statistics, these functions are often applied entrywise to the off-diagonal entries of correlation matrices to improve the quality of the correlation matrix.

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In modern high-dimensional probability and statistics, these functions are often applied entrywise to the off-diagonal entries of correlation matrices to improve the quality of the correlation matrix.

Even for sparse correlation matrices, no universal ϵ exists such that the application of f_ϵ^H or f_ϵ^S on the off-diagonal entries preserves positivity.

Loewner positive functions operating on off-diagonals

Definition Let $I \subset \mathbb{C}$, $f : I \to \mathbb{C}$, and $n \geq 1$. Define $f_*[-] : I^{n\times n} \to \mathbb{C}^{n\times n}$ as

$$f_*[A] = \begin{pmatrix} a_{11} & f(a_{12}) & f(a_{13}) & & f(a_{1n}) \\ f(a_{21}) & a_{22} & f(a_{23}) & \dots & f(a_{2n}) \\ f(a_{31}) & f(a_{32}) & a_{33} & & f(a_{3n}) \\ & \vdots & & \ddots & \vdots \\ f(a_{n1}) & f(a_{n2}) & f(a_{n3}) & \dots & a_{nn} \end{pmatrix} \qquad \text{for all } A = (a_{ij}) \in I^{n\times n}.$$

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Classification in a real setting:

Theorem (Guillot–Rajaratnam [*Trans. Amer. Math. Soc.* 2015])
Let $I = (-\rho, \rho)$ for $0 < \rho \leq \infty$, and $f : I \to \mathbb{R}$. TFAE:

1. $f_*[A] \in \mathbb{P}_n$ for all $A = (a_{ij}) \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. - $f(x) = \sum_{k\geq 0} c_k x^k$ for all $x \in I$, where $c_k \geq 0$ for all $k \geq 0$, and

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   - $f(x) = \sum_{k\geq 0} c_k x^k$ for all $x \in I$, where $c_k \geq 0$ for all $k \geq 0$, and
   - $|f(x)| \leq |x|$ for all $x \in I$.

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Two functions acting entrywise

Definition

Let $n \geq 1$ be an integer, $I \subset \mathbb{C}$ and $g, f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]}$, where $[n] := \{1, \ldots, n\}$. Define

$$(g, f)_{T_n}[-] : I^{n \times n} \to \mathbb{C}^{n \times n},$$

such that for all $A = (a_{ij}) \in I^{n \times n}$,

$$(g, f)_{T_n}[A]_{ij} := \begin{cases} g(a_{ij}) & \text{if } i, j \in U \text{ for some } U \in T_n, \\ f(a_{ij}) & \text{otherwise.} \end{cases}$$

For instance

$$T_3 = \{\{1, 3\}, \{2\}\} \quad \implies \quad (g, f)_{T_3}[A] = \begin{pmatrix} g(a_{11}) & f(a_{12}) & g(a_{13}) \\ f(a_{21}) & g(a_{22}) & f(a_{23}) \\ g(a_{31}) & f(a_{32}) & g(a_{33}) \end{pmatrix}$$

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$$T_4 = \{\{1\}, \{2, 3\}, \{2, 4\}\} \implies (g, f)_{T_4}[A] = \begin{pmatrix} g(a_{11}) & f(a_{12}) & f(a_{13}) & f(a_{14}) \\ f(a_{21}) & g(a_{22}) & g(a_{23}) & g(a_{24}) \\ f(a_{31}) & g(a_{32}) & g(a_{33}) & f(a_{34}) \\ f(a_{41}) & g(a_{42}) & f(a_{43}) & g(a_{44}) \end{pmatrix}$$

Question

Given a sequence $(T_n)_{n\geq 1}$, what are all those functions $g, f : I \to \mathbb{C}$ such that the two-function operation $(g, f)_{T_n}[-]$ preserves positivity over $\mathbb{P}_n(I)$ for all $n \geq 1$?

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Note that

- $f_{T_n}[-] = f[-]$ when $T_n = \{\emptyset\}$. *(Schoenberg, Rudin / Herz / Vasudeva)*
- $f_{T_n}[-] = f_*[-]$ when $T_n = \{\{1\}, \{2\}, \ldots, \{n\}\}$. *(Guillot–Rajaratnam)*
  - In all of these cases, the preservers are absolutely monotonic / infinite sums of Herz functions.

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  - In all of these cases, the preservers are absolutely monotonic / infinite sums of Herz functions.

Our contribution

- Classification for every other $(T_n)_{n\geq 1}$.
- This uncovers dimension-free preservers that are **not** absolutely monotonic / infinite sums of Herz functions.

	$(T_n)_{n\geq 1}$	(g,f) $I = D(0,\rho)$, where $0 < \rho \leq \infty$	(g,f) I_ρ, where $0 < \rho \leq \infty$	f $I = D(0,\rho)$, where $0 < \rho \leq \infty$	f I_ρ, where $0 < \rho \leq \infty$
1.	$T_n = \emptyset$ for all $n \geq 1$	$f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k} \geq 0$	$f(x) = \sum_{k\geq 0} c_k x^k$ where all $c_k \geq 0$	$f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k} \geq 0$	$f(x) = \sum_{k\geq 0} c_k x^k$ where all $c_k \geq 0$
2.	$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_n \neq \emptyset$ for some $n \geq 2$	$f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k} \geq 0$, $g(x) \geq f(x)$ over $I_{\geq 0}$	$f(x) = \sum_{k\geq 0} c_k x^k$ where all $c_k \geq 0$, $g(x) \geq f(x)$ over $I_{\geq 0}$	$f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k} \geq 0$, $x \geq f(x)$ over $I_{\geq 0}$	$f(x) = \sum_{k\geq 0} c_k x^k$ where all $c_k \geq 0$, $x \geq f(x)$ over $I_{\geq 0}$
3.	$T_n =$ subpartition$([n])$ for all $n \geq 3$, and $T_n \not\subseteq \{\{j\} : j \in [n]\}$ for some $n \geq 3$	for $g(z) = \alpha z^m \overline{z}^k$ where $\alpha \geq 0, m, k \in \mathbb{Z}_{\geq 0}$: $f(z) = cg(z)$, where	for $g(z) = \alpha x^k$ where $\alpha \geq 0, k \in \mathbb{Z}_{\geq 0}$: $f(x) = cg(x)$, where	$f(z) = cz$, where	$f(x) = cx$, where
3.a	$\sqcup_{J\in T_n} J = [n]$ for all $n \geq 1$, and $K := \max_{n\geq 1} \lvert T_n \rvert \in \mathbb{Z}$	$c \in [-1/(K-1), 1]$	$c \in [-1/(K-1), 1]$	$c \in [-1/(K-1), 1]$	$c \in [-1/(K-1), 1]$
3.b	remaining sub-cases	$c \in [0,1]$	$c \in [0,1]$	$c \in [0,1]$	$c \in [0,1]$
4.	$T_n \neq$ subpartition$([n])$ for some $n \geq 3$	$f(z) = g(z) =$ $\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$, where $c_{m,k} \geq 0$	$f(x) = g(x) =$ $\sum_{k\geq 0} c_k x^k$, where $c_k \geq 0$	$f(z) = z$ (over any $I \subseteq \mathbb{C}$)	$f(x) = x$ (over any $I \subseteq \mathbb{R}$)

TABLE 2. $(T_n)_{n\geq 1}$ against (g,f) and f for domains $D(0,\rho)$ and I_ρ, where I_ρ is any of the real domains $(-\rho,\rho), (0,\rho)$ and $[0,\rho)$. 'Subpartition$([n])$' here refers to a partition of a subset of $[n]$. Here we study the dimension free case, i.e. assume for all $n \geq 2$ that $T_n \neq \{[n]\}$; and for each n, the subsets in T_n are pairwise incomparable.

PART 3

The classifications.

Main reference

P.V., Positivity preservers forbidden to operate on diagonal blocks.
Trans. Amer. Math. Soc., 376, pp. 5261-5279, 2023.

Theorems A, B, and C

Let $I = D(0,\rho)$ *for* $0 < \rho \leq \infty$, *and* $g, f : I \to \mathbb{C}$. *Let* $T_n \subseteq 2^{[n]}$ *for all* $n \geq 1$ *such that at least one* T_n *is nonempty.*

Theorem A

$$(g, f)_{T_n}[A] = \begin{pmatrix} g(a_{11}) & f(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ f(a_{21}) & g(a_{22}) & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & g(a_{44}) & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$, and $g, f : I \to \mathbb{C}$. Let $T_n \subseteq 2^{[n]}$ for all $n \geq 1$ such that at least one T_n is nonempty.

- Suppose $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$.

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- Suppose $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$.

TFAE.

1. $(g,f)_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. - $f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m,k \geq 0$.
   - $g(z) \geq f(z)$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

- Two-fold generalization of Guillot–Rajaratnam (*Trans. Amer. Math. Soc.* 2015).

Theorem B

$$(g,f)_{T_n}[A] = \begin{pmatrix} g(a_{11}) & g(a_{12}) & f(a_{13}) & f(a_{14}) & \dots \\ g(a_{21}) & g(a_{22}) & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \dots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & g(a_{44}) & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n\times n}$$

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$, and $g, f : I \to \mathbb{C}$. Let $T_n \subseteq 2^{[n]}$ for all $n \geq 1$ such that at least one T_n is nonempty.

- Suppose
  - each T_n is a partition of a *subset* of $[n]$, and
  - there exists $N \geq 3$ such that $T_N \not\subseteq \{\{j\} : j \in [N]\}$.

Theorem B

$$(g,f)_{T_n}[A] = \begin{pmatrix} g(a_{11}) & g(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ g(a_{21}) & g(a_{22}) & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & g(a_{44}) & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n\times n}$$

Let $I = D(0,\rho)$ *for* $0 < \rho \leq \infty$, *and* $g, f : I \to \mathbb{C}$. *Let* $T_n \subseteq 2^{[n]}$ *for all* $n \geq 1$ *such that at least one* T_n *is nonempty.*

- Suppose
  - each T_n is a partition of a *subset* of $[n]$, and
  - there exists $N \geq 3$ such that $T_N \not\subseteq \{\{j\} : j \in [N]\}$.
- Suppose g is a Herz function, i.e., $g(z) := \alpha z^m \overline{z}^k$ for $\alpha \geq 0$ and integer $m, k \geq 0$.

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Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$, and $g, f : I \to \mathbb{C}$. Let $T_n \subseteq 2^{[n]}$ for all $n \geq 1$ such that at least one T_n is nonempty.

- Suppose
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TFAE.

1. $(g, f)_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. $f(z) = cg(z)$ for all $z \in I$, where,
   - **a.** $c \in [-1/(K-1), 1]$ — if T_n is a partition of $[n]$ for all $n \geq 1$, and $K := \max_{n \geq 1} |T_n| < \infty$.
   - **b.** $c \in [0, 1]$ for the remaining cases.

Theorem C

$$(g, f)_{T_n}[A] = \begin{pmatrix} g(a_{11}) & g(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ g(a_{21}) & g(a_{22}) & g(a_{23}) & f(a_{24}) & \\ f(a_{31}) & g(a_{32}) & g(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & g(a_{44}) & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$, and $g, f : I \to \mathbb{C}$. Let $T_n \subseteq 2^{[n]}$ for all $n \geq 1$ such that at least one T_n is nonempty.

- Suppose there exists $N \geq 3$ such that T_N is not a partition of **any** subset of $[N]$.

Theorem C

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Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$, and $g, f : I \to \mathbb{C}$. Let $T_n \subseteq 2^{[n]}$ for all $n \geq 1$ such that at least one T_n is nonempty.

- Suppose there exists $N \geq 3$ such that T_N is not a partition of **any** subset of $[N]$.

TFAE.

1. $(g, f)_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. $g(z) = f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.

Theorem A	Theorem B	Theorem C
g acts on diagonal entries then $f \leq g$	g acts on off-diagonal entries with no overlap then $f = cg$ with $\lvert c\rvert \leq 1$	g acts on off-diagonal entries with overlap then $f = g$

Table 2.2: Observe the decreasing difference between g and f as the size of blocks where g operates grows.

Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. Exactly one of the following holds:

   S. If $T_n = \emptyset$ for all $n \geq 1$, then $\rightsquigarrow$ Schoenberg, Rudin / Herz / Vasudeva

   - $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.

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   A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then

   - $f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m,k \geq 0$.
   - $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$. $\rightsquigarrow$ Guillot–Rajaratnam (real)

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Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

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A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then

- $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
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B. If T_n is a partition of a subset of $[n]$ for all $n \geq 1$, and there exists $N \geq 3$ such that there is a $U \in T_N$ with $|U| \geq 2$, then

$$f(z) = cz \text{ for all } z \in I$$

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   B. If T_n is a partition of a subset of $[n]$ for all $n \geq 1$, and there exists $N \geq 3$ such that there is a $U \in T_N$ with $|U| \geq 2$, then

   $$f(z) = cz \text{ for all } z \in I, \text{ where}$$

   a. $c \in [-1/(K-1), 1]$, where $\max_{n\geq 1} |T_n| =: K < \infty$, and each T_n is a partition of $[n]$.

Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

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A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then

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- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$. $\rightsquigarrow$ Guillot–Rajaratnam (real)

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a. $c \in [-1/(K-1), 1]$, where $\max_{n\geq 1} |T_n| =: K < \infty$, and each T_n is a partition of $[n]$.

b. $c \in [0, 1]$ for the remaining cases.

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b. $c \in [0,1]$ for the remaining cases.

C. If T_N is not a partition of any subset of $[N]$ for some $N \geq 3$, then

$$f(z) = z \text{ for all } z \in I.$$

f acts on all the entries $\Updownarrow$ f is an infinite sum of Herz functions	f is forbidden from some 1×1 principal block $\Updownarrow$ f is an infinite sum of Herz functions, which is pointwise bounded-above by the identity function	f is forbidden from some $k \times k$ principal block, $k \geq 2$ $\Updownarrow$ f is linear, vanishing at the origin, and sometimes with negative slope

Table 2.1: Observe the contrast in the class of the positivity preservers when the size of the forbidden principal block is increased from 1×1 to $k \times k$, for $k \geq 2$.

PART 4

Proofs.

Folklore results on matrices

(a) (Weyl's inequality.) Let $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$, respectively, denote the maximum and minimum eigenvalues of square matrix X. Then, for $n \times n$ Hermitian matrices A and B,

$$\lambda_{\min}(A) + \lambda_{\max}(B) \geq \lambda_{\min}(A+B) \geq \lambda_{\min}(A) + \lambda_{\min}(B).$$

Note that $\lambda_{\min}(X)$ is super-additive over the class of Hermitian matrices X.

(b) (Eigen-pairs of the tensor product.) Let $A_{n\times n}$ and $B_{m\times m}$ be two Hermitian matrices. Suppose

$$\begin{aligned} A \text{ has eigen-pairs } \big(\lambda_i(A),\ v_i(A)\big), &\qquad 1 \leq i \leq n, \text{ and} \\ B \text{ has eigen-pairs } \big(\lambda_j(B),\ v_j(B)\big), &\qquad 1 \leq j \leq m. \end{aligned}$$

Then,

$$A \otimes B \text{ has eigen-pairs } \big(\lambda_i(A)\lambda_j(B),\ v_i(A) \otimes v_j(B)\big), \quad \begin{array}{l} 1 \leq i \leq n, \\ 1 \leq j \leq m. \end{array}$$

Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.
2. Exactly one of the following holds:

S. If $T_n = \emptyset$ for all $n \geq 1$, then $\rightsquigarrow$ Schoenberg, Rudin / Herz / Vasudeva

- $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m,k \geq 0$.

A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then

- $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m,k \geq 0$.
- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$. $\rightsquigarrow$ Guillot–Rajaratnam (real)

B. If T_n is a partition of a subset of $[n]$ for all $n \geq 1$, and there exists $N \geq 3$ such that there is a $U \in T_N$ with $|U| \geq 2$, then

$$f(z) = cz \text{ for all } z \in I, \text{ where}$$

a. $c \in [-1/(K-1), 1]$, where $\max_{n \geq 1} |T_n| =: K < \infty$, and each T_n is a partition of $[n]$.

b. $c \in [0,1]$ for the remaining cases.

C. If T_N is not a partition of any subset of $[N]$ for some $N \geq 3$, then

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Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then
   - $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
   - $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

(A) $\implies$ (1) (sketch)

$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$.

$$f_{T_n}[A] = \begin{pmatrix} a_{11} & f(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ f(a_{21}) & a_{22} & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & a_{44} & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

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1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

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- $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

Observe that,

$$f_{T_n}[A] = f[A] + \mathbf{0}_{T_n}[A - f[A]], \qquad \text{for all } A \in \mathbb{P}_n(I), \text{ for all } n \geq 1.$$

(A) $\implies$ (1) (sketch)

$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$.

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- $f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

Observe that,

$$f_{T_n}[A] = f[A] + \mathbf{0}_{T_n}[A - f[A]], \qquad \text{for all } A \in \mathbb{P}_n(I), \text{ for all } n \geq 1.$$

- $f[A]$ is PSD by the Herz theorem.
- Since, $z \geq f(z)$ for all $z \in I \cap \mathbb{R}_{\geq 0}$, $\mathbf{0}_{T_n}[A - f[A]]$ is PSD.

(A) $\implies$ (1) (sketch)

$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$.

$$f_{T_n}[A] = \begin{pmatrix} a_{11} & f(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ f(a_{21}) & a_{22} & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & a_{44} & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

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A. The following holds:

- $f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

Observe that,

$$f_{T_n}[A] = f[A] + \mathbf{0}_{T_n}[A - f[A]], \qquad \text{for all } A \in \mathbb{P}_n(I), \text{ for all } n \geq 1.$$

- $f[A]$ is PSD by the Herz theorem.
- Since, $z \geq f(z)$ for all $z \in I \cap \mathbb{R}_{\geq 0}$, $\mathbf{0}_{T_n}[A - f[A]]$ is PSD.

Therefore, $f_{T_n}[A] \in \mathbb{P}_n$. This completes the implication.

$(1) \implies (A)$ (sketch)

$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$.

$$f_{T_n}[A] = \begin{pmatrix} a_{11} & f(a_{12}) & f(a_{13}) & f(a_{14}) & \cdots \\ f(a_{21}) & a_{22} & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & f(a_{33}) & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & a_{44} & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

(1) $\implies$ (A) (sketch)

$T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$.

$$\begin{pmatrix} f(a_{11}) & f(a_{12}) & \cdots & f(a_{1n}) \\ f(a_{21}) & f(a_{22}) & \cdots & f(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_{n1}) & f(a_{n2}) & \cdots & f(a_{nn}) \end{pmatrix}_{n\times n}$$

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1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

A. The following holds:
- $f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
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- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

- If $f[A]$ is PSD for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$, then invoke Herz theorem.
- If $f_*[A]$ is PSD for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$, then we do the following.

(1) $\implies$ (A) (sketch, cont.)

$$\begin{aligned} f_*[\mathbf{1}_m \otimes A] &= f[\mathbf{1}_m \otimes A] + \mathrm{Id}_m \otimes \mathrm{diag}(a_{ii} - f(a_{ii})) \\ &= \mathbf{1}_m \otimes f[A] + \mathrm{Id}_m \otimes \mathrm{diag}(a_{ii} - f(a_{ii})). \end{aligned}$$

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$$\begin{aligned} f_*[\mathbf{1}_m \otimes A] &= f[\mathbf{1}_m \otimes A] + \mathrm{Id}_m \otimes \mathsf{diag}(a_{ii} - f(a_{ii})) \\ &= \mathbf{1}_m \otimes f[A] + \mathrm{Id}_m \otimes \mathsf{diag}(a_{ii} - f(a_{ii})). \end{aligned}$$

Using Weyl's inequality for Hermitian matrices,

$$\begin{aligned} 0 \leq \lambda_{\min}(f_*[\mathbf{1}_m \otimes A]) &= \lambda_{\min}(\mathbf{1}_m \otimes f[A] + \mathrm{Id}_m \otimes \mathsf{diag}(a_{ii} - f(a_{ii}))) \\ &\leq \lambda_{\min}(\mathbf{1}_m \otimes f[A]) + \lambda_{\max}(\mathrm{Id}_m \otimes \mathsf{diag}(a_{ii} - f(a_{ii}))) \\ &\leq m\lambda_{\min}(f[A]) + \max_{i \in [1,n]} (a_{ii} - f(a_{ii})). \end{aligned}$$

$(\mathbf{1}) \implies (\mathbf{A})$ (sketch, cont.)

$$\begin{aligned} f_*[\mathbf{1}_m \otimes A] &= f[\mathbf{1}_m \otimes A] + \mathrm{Id}_m \otimes \mathrm{diag}(a_{ii} - f(a_{ii})) \\ &= \mathbf{1}_m \otimes f[A] + \mathrm{Id}_m \otimes \mathrm{diag}(a_{ii} - f(a_{ii})). \end{aligned}$$

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This gives us,

$$\lambda_{\min}(f[A]) \geq -\frac{1}{m} \max_{i \in [1,n]} (a_{ii} - f(a_{ii})).$$

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This gives us,

$$\lambda_{\min}(f[A]) \geq -\frac{1}{m} \max_{i \in [1,n]} (a_{ii} - f(a_{ii})).$$

Since m can be arbitrarily large,

$$\lambda_{\min}(f[A]) \geq 0.$$

Invoke Herz theorem to conclude,

$$f(z) = \sum_{m,k \geq 0} c_{m,k} z^m \overline{z}^k \text{ for all } z \in I, \text{ where } c_{m,k} \geq 0.$$

(1) $\implies$ (A) (sketch, cont.)

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- $f(z) \leq z$ for all $z \in I \cap \mathbb{R}_{\geq 0}$.

Since $T_N \neq \emptyset$ for some $N \geq 2$, by the positive semidefiniteness of

$$\begin{pmatrix} z & f(z) \\ f(z) & z \end{pmatrix} \text{ or } \begin{pmatrix} z & f(z) \\ f(z) & f(z) \end{pmatrix},$$

we have $f(z) \leq z$, for $z \in I \cap \mathbb{R}_{\geq 0}$.

Corollary D

Let $I = D(0,\rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

A. If $T_n \subseteq \{\{j\} : j \in [n]\}$ for all $n \geq 1$, and $T_N \neq \emptyset$ for some $N \geq 2$, then

- $f(z) = \sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ for all $z \in I$, where $c_{m,k} \geq 0$ for all $m, k \geq 0$.
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Corollary D

Let $I = D(0, \rho)$ for $0 < \rho \leq \infty$; $f : I \to \mathbb{C}$. Suppose $T_n \subset 2^{[n]} \setminus \{[n]\}$, $n \geq 1$. TFAE:

1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

B. If T_n is a partition of a subset of $[n]$ for all $n \geq 1$, and there exists $N \geq 3$ such that there is a $U \in T_N$ with $|U| \geq 2$, then

$$f(z) = cz \text{ for all } z \in I, \text{ where}$$

a. $c \in [-1/(K-1), 1]$, where $\max_{n \geq 1} |T_n| = K \in \mathbb{Z}$, if each T_n is a partition of $[n]$.

(1) $\iff$ (B.a)

$$f_{T_n}[A] = \begin{pmatrix} a_{11} & a_{12} & f(a_{13}) & f(a_{14}) & \cdots \\ a_{21} & a_{22} & f(a_{23}) & f(a_{24}) & \\ f(a_{31}) & f(a_{32}) & a_{33} & f(a_{34}) & \cdots \\ f(a_{41}) & f(a_{42}) & f(a_{43}) & a_{44} & \\ \vdots & & \vdots & & \ddots \end{pmatrix}_{n \times n}$$

(1) $\implies$ (B.a) (sketch)

Each T_n is a partition of $[n]$ with $2 \leq |T_n| \leq n-1$.

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Each T_n is a partition of $[n]$ with $2 \le |T_n| \le n-1$. Set $K := \max_{n\ge1} |T_n| < \infty$.

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- There exists $n_1 \geq 3$ such that $T_{n_1} = \{\{u_1, u_2, \dots\}, \{u_3, \dots\}, \dots\}$.

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(1) $\Longrightarrow$ (B.a) (sketch)

Each T_n is a partition of $[n]$ with $2 \leq |T_n| \leq n-1$. Set $K := \max_{n \geq 1} |T_n| < \infty$.

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1. $f_{T_n}[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$, for all $n \geq 1$.

Step 1. Observe that, for $|z| \leq w < \rho$,
$$\begin{pmatrix} |z|^2/w & z & z & \mathbf{0} \\ \overline{z} & w & w & \dots \\ \overline{z} & w & w & \\ \mathbf{0}^T & \vdots & & \ddots \end{pmatrix}_{n_1 \times n_1} \in \mathbb{P}_{n_1}(I).$$

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Leading 3×3 minor is nonnegative, which needs

$$-\left(z\frac{f(w)}{\sqrt{w}} - \sqrt{w} f(z)\right)^2 \geq 0,$$

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Leading 3×3 minor is nonnegative, which needs

$$-\left(z\frac{f(w)}{\sqrt{w}} - \sqrt{w}f(z)\right)^2 \geq 0,$$

and therefore

$$f(z) = \frac{f(w)}{w}z \qquad \text{whenever} \qquad |z| \leq w < \rho.$$

Now conclude that

$$f(z) = cz \text{ for all } z \in I, \text{ and for some } c \in [-1,1].$$

(1) $\implies$ (B.a) (sketch, cont.)

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For $0 < |z| \in I$,

$$f_*[|z|\mathbf{1}_{K\times K}] = \begin{pmatrix} |z| & c|z| & c|z| & \\ c|z| & |z| & c|z| & \dots \\ c|z| & c|z| & |z| & \\ & \vdots & & \ddots \end{pmatrix}_{K\times K} \quad \text{has spectrum} \quad \begin{cases} \lambda_1 = & (1-c)|z|, \\ \lambda_2 = & (1+(K-1)c)|z|. \end{cases}$$

$(1) \implies$ (B.a) (sketch, cont.)

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$$f_*[|z|\mathbf{1}_{K\times K}] = \begin{pmatrix} |z| & c|z| & c|z| & \\ c|z| & |z| & c|z| & \cdots \\ c|z| & c|z| & |z| & \\ & \vdots & & \ddots \end{pmatrix}_{K\times K} \quad \text{has spectrum} \quad \begin{cases} \lambda_1 = & (1-c)|z|, \\ \lambda_2 = & (1+(K-1)c)|z|. \end{cases}$$

Note that this is a principal submatrix of $f_{T_{n_2}}[|z|\mathbf{1}_{n\times n}]$ (corresponding to indices $u_1, u_2, \dots, u_K$).

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For $0 < |z| \in I$,

$$f_*[|z|\mathbf{1}_{K\times K}] = \begin{pmatrix} |z| & c|z| & c|z| & \\ c|z| & |z| & c|z| & \dots \\ c|z| & c|z| & |z| & \\ & \vdots & & \ddots \end{pmatrix}_{K\times K} \quad \text{has spectrum} \quad \begin{cases} \lambda_1 = & (1-c)|z|, \\ \lambda_2 = & (1+(K-1)c)|z|. \end{cases}$$

Note that this is a principal submatrix of $f_{T_{n_2}}[|z|\mathbf{1}_{n\times n}]$ (corresponding to indices $u_1, u_2, \dots, u_K$). As this is positive, we deduce:

$$c \in [-1/(K-1), 1].$$

(B.a) $\implies$ (1) (sketch)

Recall each T_n is a partition of $[n]$ with $2 \leq |T_n| \leq n-1$. Set $K := \max_{n\geq 1} |T_n| < \infty$.

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Observe that

$$f_{T_n}[A] = A \circ f_{T_n}[\mathbf{1}_{n \times n}] \qquad \text{for all } A \in \mathbb{P}_n(I), \text{ for all } n \ge 1.$$

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- The principal submatrices of $f_{T_n}[\mathbf{1}_{n\times n}]$ that are of the form $f_*[\mathbf{1}_{k\times k}]$ where $k \leq K$ have nonnegative eigenvalues $\begin{cases} \lambda_1 = & 1-c, \\ \lambda_2 = & 1+(k-1)c. \end{cases}$ (We saw this in Step 2.)

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- Its remaining submatrices have two identical rows, therefore their minor vanishes.

This completes the sketch of the equivalence.

		(g,f)	(g,f)	f	f
	$(T_n)_{n\geq 1}$	$I = D(0,\rho)$, where $0<\rho\leq\infty$	I_ρ, where $0<\rho\leq\infty$	$I = D(0,\rho)$, where $0<\rho\leq\infty$	I_ρ, where $0<\rho\leq\infty$
1.	$T_n = \emptyset$ for all $n\geq 1$	$f(z)=\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k}\geq 0$	$f(x)=\sum_{k\geq 0} c_k x^k$ where all $c_k\geq 0$	$f(z)=\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k}\geq 0$	$f(x)=\sum_{k\geq 0} c_k x^k$ where all $c_k\geq 0$
2.	$T_n \subseteq \{\{j\} : j\in[n]\}$ for all $n\geq 1$, and $T_n\neq\emptyset$ for some $n\geq 2$	$f(z)=\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k}\geq 0$, $g(x)\geq f(x)$ over $I_{\geq 0}$	$f(x)=\sum_{k\geq 0} c_k x^k$ where all $c_k\geq 0$, $g(x)\geq f(x)$ over $I_{\geq 0}$	$f(z)=\sum_{m,k\geq 0} c_{m,k} z^m \overline{z}^k$ where all $c_{m,k}\geq 0$, $x\geq f(x)$ over $I_{\geq 0}$	$f(x)=\sum_{k\geq 0} c_k x^k$ where all $c_k\geq 0$, $x\geq f(x)$ over $I_{\geq 0}$
3.	$T_n =$ subpartition$([n])$ for all $n\geq 3$, and $T_n\not\subseteq\{\{j\} : j\in[n]\}$ for some $n\geq 3$	for $g(z)=\alpha z^m\overline{z}^k$ where $\alpha\geq 0, m,k\in\mathbb{Z}_{\geq 0}$: $f(z)=cg(z)$, where	for $g(z)=\alpha x^k$ where $\alpha\geq 0, k\in\mathbb{Z}_{\geq 0}$: $f(x)=cg(x)$, where	$f(z)=cz$, where	$f(x)=cx$, where
3.a	$\sqcup_{J\in T_n} J = [n]$ for all $n\geq 1$, and $K := \max_{n\geq 1}\lvert T_n\rvert\in\mathbb{Z}$	$c\in[-1/(K-1),1]$	$c\in[-1/(K-1),1]$	$c\in[-1/(K-1),1]$	$c\in[-1/(K-1),1]$
3.b	remaining sub-cases	$c\in[0,1]$	$c\in[0,1]$	$c\in[0,1]$	$c\in[0,1]$
4.	$T_n\neq$ subpartition$([n])$ for some $n\geq 3$	$f(z)=g(z)=\sum_{m,k\geq 0} c_{m,k} z^m\overline{z}^k$, where $c_{m,k}\geq 0$	$f(x)=g(x)=\sum_{k\geq 0} c_k x^k$, where $c_k\geq 0$	$f(z)=z$ (over any $I\subseteq\mathbb{C}$)	$f(x)=x$ (over any $I\subseteq\mathbb{R}$)

TABLE 2. $(T_n)_{n\geq 1}$ against (g,f) and f for domains $D(0,\rho)$ and I_ρ, where I_ρ is any of the real domains $(-\rho,\rho),(0,\rho)$ and $[0,\rho)$. 'Subpartition$([n])$' here refers to a partition of a subset of $[n]$. Here we study the dimension free case, i.e. assume for all $n\geq 2$ that $T_n\neq\{[n]\}$; and for each n, the subsets in T_n are pairwise incomparable.

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- Increases the richness of the preserver problem by:
  - introducing entrywise variants (g, f) preserving positivity, and
  - their classification for real and complex domains.

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