Open Quantum Systems, Quantum Tomography

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Ongoing project with

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Introduction for open quantum system

- For a closed system, quantum states are represented by unit vectors in a Hilbert space, say, $\mathbb{C}^n$ with inner product

  $\langle x|y\rangle = \sum_{j=1}^{n} \bar{x}_j y_j$ if $|x\rangle = (x_1, \ldots, x_n)^t, |y\rangle = (y_1, \ldots, y_n)^t$.

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- If the measurement device will give a reading λ_j when $|x\rangle$ collapses to $|u_j\rangle$, then the expectation of the measurement of $|x\rangle$ will be

$$\sum_{j=1}^n \lambda_j |\langle u_j|x_j\rangle|^2 = \text{tr}\,(A|x\rangle\langle x|) = \langle x|A|x\rangle,$$

where $A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^*$ and U has columns $|u_1\rangle, \ldots, |u_n\rangle$.

If $A = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$, then $\langle x|A|x\rangle = \sum_{j=1}^n \lambda_j |x_j|^2$.

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- The evolution of the system is governed by the Schrödinger equation, and we have $|x(t)\rangle = e^{iH_t}|x(0)\rangle = V_t|x(0)\rangle$, where V_t is unitary.

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- If $\rho = |x\rangle\langle x|$ for a unit vector $|x\rangle$, then ρ a pure state.

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- If we consider $U|\psi\rangle\langle\psi|U^\dagger$ in the principal system we can apply the linear map tr_2 such that $\operatorname{tr}_2(\rho_1 \otimes \rho_2) = \rho_1$ for any $\rho_1 \in M_p$ and $\rho_2 \in M_q$.

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- We can then determine $(1+b)/2, (1+c)/2$, etc.

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- In general, to determine a density matrix $\rho \in M_N$, we need to determine
  * N real diagonal entries summing up to 1,
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- But we need to find $U_0, \ldots, U_N$ that can be implemented efficiently, and the measurements can be obtained accurately.

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Theorem [Li, Nakahara, Pelejo, Stanish, Wang (2023?)]

An n-qubit state ρ can be determined by the measurements of $U_j \rho U_j^*$ for $j = 1, \ldots, 3^n$, where $U_j = V_{j1} \otimes \cdots \otimes V_{jn}$ with

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Recall that for $\rho \in M_N$ with 2^n, we can find $N + 1 = 2^n + 1$ unitary matrices $U_0, \ldots, U_N \in M_N$ to determine ρ. Here, we need 3^n local unitary matrices.

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- Technically, one needs to construct a quantum operation $\Phi : M_N \to M_{N^2}$ and uses the diagonal entries of $\Phi(\rho)$ to determine ρ.

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- For example, to determine $\rho \in M_{2^2}$, we construct $\tilde{\rho} \in M_{2^4}$ so that the 16 diagonal entries of $\tilde{\rho}$ can be used to determine ρ.
- In general, one can entangle an n-qubit state $\rho \in M_N$ with another n-qubit state to get a $2n$-qubit state $\hat{\rho} \in M_{N^2}$ so that the N^2 diagonal entries of $\hat{\rho}$ can be used to determine ρ.
- For $\rho \in M_N$, one need to determine ρ using the diagonal entries of

$$U(E_{11} \otimes \rho)U^* = U \begin{pmatrix} \rho & \\ & 0_{N^2-N} \end{pmatrix} U^* \in M_{N^2}.$$

- Technically, one needs to construct a quantum operation $\Phi : M_N \to M_{N^2}$ and uses the diagonal entries of $\Phi(\rho)$ to determine ρ.
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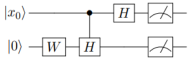
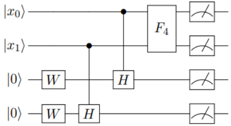
- Showing the existence of such a map is non-trivial.
- Moreover, one needs to find an efficient method to do the construction.

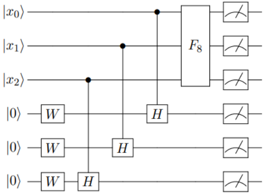
- **1-qubit Assisted State Tomography**



- **2-qubit Assisted State Tomography**



- **3-qubit Assisted State Tomography**



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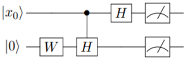
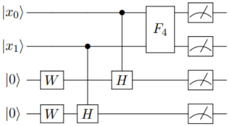


- **2-qubit Assisted State Tomography**



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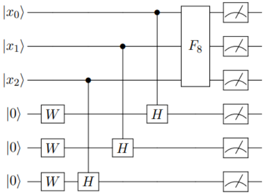

Theorem [LNPSW (2023?)]

Let $\rho \in M_N$ with $N = 2^n$. Then ρ can be determined by the diagonal entries of the matrix $U(E_{11} \otimes \rho)U^* \in M_{N^2}$ for

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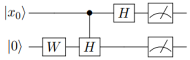
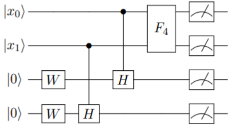


- **2-qubit Assisted State Tomography**



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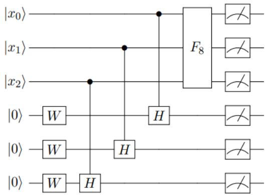

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where $W = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3}i \\ \sqrt{3}i & 1 \end{pmatrix}$, F_n is the discrete Fourier transform, and C_j is the controlled Hadamard gate such that $C_j|q_{2n-1}, \ldots, q_0\rangle$ will change

$$|q_{2n-j}\rangle \text{ to } H|q_{2n-j}\rangle \text{ if } |q_{n-j}\rangle = |1\rangle \text{ for } j = 0, \ldots, n-1.$$

Further research

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- There are much to explored, say, using other quantum computing devices such as NMR, trapped ions, optics, etc. to do tomography.
- After getting the measurements one need to use numerical methods (build a maximum likelihood function), statistical methods, symmetries in physics laws, and AI to help find the best quantum states producing the measurement data.

Thank you for your attention!

Your comments are welcome!