Matrix problems in Quantum Computing

Chi-Kwong Li
Department of Mathematics, The College of William and Mary;
Institute for Quantum Computing, University of Waterloo

Quantum postulates - Superposition

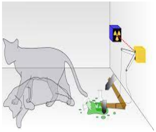
- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.

Quantum postulates - Superposition

- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.
- Example. Photons can be in vertical and horizontal polarization, represented by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Quantum postulates - Superposition

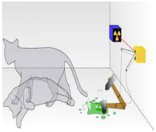
- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.
- Example. Photons can be in vertical and horizontal polarization, represented by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- A photon state has the form $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$.



Think about the Schrödinger cat.

Quantum postulates - Superposition

- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.
- Example. Photons can be in vertical and horizontal polarization, represented by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- A photon state has the form $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$.



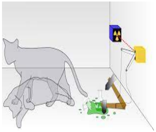
Think about the Schrödinger cat.

- A 2-phonton state is a unit vector $|\psi\rangle \in \mathbb{C}^4$ represented as a linear combination of

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Quantum postulates - Superposition

- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.
- Example. Photons can be in vertical and horizontal polarization, represented by $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.
- A photon state has the form $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix}a\\b\end{pmatrix}$ with $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$.



Think about the Schrödinger cat.

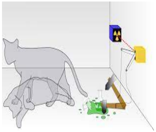
- A 2-phonton state is a unit vector $|\psi\rangle \in \mathbb{C}^4$ represented as a linear combination of

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \ |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \ |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \ |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

- An n-photon state is a unit vector $|\psi\rangle$ in $\mathbb{C}^N$ with $N = 2^n$ represented as a linear combination of $|0\ldots0\rangle, \ldots, |1\cdots1\rangle$.

Quantum postulates - Superposition

- Quantum states are unit vectors in a Hilbert space represented as linear combinations of a fixed orthonormal basis of the state space.
- Example. Photons can be in vertical and horizontal polarization, represented by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- A photon state has the form $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$.



Think about the Schrödinger cat.

- A 2-phonton state is a unit vector $|\psi\rangle \in \mathbb{C}^4$ represented as a linear combination of

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- An n-photon state is a unit vector $|\psi\rangle$ in $\mathbb{C}^N$ with $N = 2^n$ represented as a linear combination of $|0 \ldots 0\rangle, \ldots, |1 \cdots 1\rangle$.
- For a small n, say, $n = 100$, the state space has very high dimension.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.
- When you open the box containing the Schrödinger cat, you will only see a alive cat $|0\rangle$ or a dead cat $|1\rangle$.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.
- When you open the box containing the Schrödinger cat, you will only see a alive cat $|0\rangle$ or a dead cat $|1\rangle$.
- When you open the box containing n Schrödinger cats, you will only see one of the states $|q_{n-1} \cdots q_0\rangle$.

The measurement process is highly discontinuous.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.
- When you open the box containing the Schrödinger cat, you will only see a alive cat $|0\rangle$ or a dead cat $|1\rangle$.
- When you open the box containing n Schrödinger cats, you will only see one of the states $|q_{n-1} \cdots q_0\rangle$. The measurement process is highly discontinuous.
- As a function of time t, a quantum system $|\psi(t)\rangle$ is governed by the Schrödinger equation

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(0)\rangle.$$

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.
- When you open the box containing the Schrödinger cat, you will only see a alive cat $|0\rangle$ or a dead cat $|1\rangle$.
- When you open the box containing n Schrödinger cats, you will only see one of the states $|q_{n-1} \cdots q_0\rangle$. The measurement process is highly discontinuous.
- As a function of time t, a quantum system $|\psi(t)\rangle$ is governed by the Schrödinger equation

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(0)\rangle.$$

- The matrix H is a Hermitian operator known as the Hamiltonian of the system, and may depends on t; the quantity $\hbar$ is the Planck's constant.

Quantum measurement, and dynamics

- Upon measurement by the basis $\{|0\rangle, |1\rangle\}$, the quantum state $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ will become $|0\rangle$ or $|1\rangle$ with probabilities $|a|^2$ and $|b|^2$, respectively.
- Measurements will "destroy" the quantum states.
- When you open the box containing the Schrödinger cat, you will only see a alive cat $|0\rangle$ or a dead cat $|1\rangle$.
- When you open the box containing n Schrödinger cats, you will only see one of the states $|q_{n-1} \cdots q_0\rangle$. The measurement process is highly discontinuous.
- As a function of time t, a quantum system $|\psi(t)\rangle$ is governed by the Schrödinger equation

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(0)\rangle.$$

- The matrix H is a Hermitian operator known as the Hamiltonian of the system, and may depends on t; the quantity $\hbar$ is the Planck's constant.
- The solution is approximated by the evolution operator U, which is unitary, such that

$$|\psi(t)\rangle = U|\psi(0)\rangle \qquad \text{with } U = e^{iH}.$$

Quantum Computing - Circuit Model

- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1} \dots q_0\rangle$ corresponding to n-digit binary numbers.

Quantum Computing - Circuit Model

- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1} \ldots q_0\rangle$ corresponding to n-digit binary numbers.
- Apply a suitable unitary operator to get $|\psi\rangle = U|\psi_0\rangle$.

Quantum Computing - Circuit Model

- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1} \ldots q_0\rangle$ corresponding to n-digit binary numbers.
- Apply a suitable unitary operator to get $|\psi\rangle = U|\psi_0\rangle$.
- Measure $U|\psi\rangle$ to get the desired result (with high probability).

Quantum Computing - Circuit Model

- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1}\dots q_0\rangle$ corresponding to n-digit binary numbers.
- Apply a suitable unitary operator to get $|\psi\rangle = U|\psi_0\rangle$.
- Measure $U|\psi\rangle$ to get the desired result (with high probability).
- For example, to determine whether a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is constant or balanced, i.e., $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ have the same size.

Quantum Computing - Circuit Model

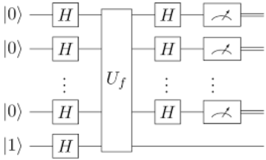
- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1}\dots q_0\rangle$ corresponding to n-digit binary numbers.
- Apply a suitable unitary operator to get $|\psi\rangle = U|\psi_0\rangle$.
- Measure $U|\psi\rangle$ to get the desired result (with high probability).
- For example, to determine whether a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is constant or balanced, i.e., $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ have the same size.

  * Classical computing require checking $f(i)$ for $2^{n-1}+1$ values of i to be sure.

- Encode a problem using n-qubits $|\psi_0\rangle = |q_{n-1} \dots q_0\rangle$ corresponding to n-digit binary numbers.
- Apply a suitable unitary operator to get $|\psi\rangle = U|\psi_0\rangle$.
- Measure $U|\psi\rangle$ to get the desired result (with high probability).
- For example, to determine whether a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is constant or balanced, i.e., $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ have the same size.

  * Classical computing require checking $f(i)$ for $2^{n-1} + 1$ values of i to be sure.

  * Deutsch-Jozsa Algorithm only requires the construction of a unitary U_f corresponding to f, and then apply it to $|\psi_0\rangle$ so that the measurement of $U_f|\psi_0\rangle$ will give the answer with high probability.



Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.
- For $H_t = (1-t)H_0 + tH_1$ for $t \in [0,1]$, let $|\psi_t\rangle$ be the ground state of H_t, i.e., the eigenvector corresponding to the smallest eigenvalue.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.
- For $H_t = (1-t)H_0 + tH_1$ for $t \in [0,1]$, let $|\psi_t\rangle$ be the ground state of H_t, i.e., the eigenvector corresponding to the smallest eigenvalue.
- Under suitable condition, the ground state $(1,\ldots,1)^t/\sqrt{2^n}$ of H_0 will change continuously to the ground state $|e_j\rangle$ of H_1.

  We can then determine the smallest diagonal entry of H_1.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.
- For $H_t = (1-t)H_0 + tH_1$ for $t \in [0,1]$, let $|\psi_t\rangle$ be the ground state of H_t, i.e., the eigenvector corresponding to the smallest eigenvalue.
- Under suitable condition, the ground state $(1,\dots,1)^t/\sqrt{2^n}$ of H_0 will change continuously to the ground state $|e_j\rangle$ of H_1.

  We can then determine the smallest diagonal entry of H_1.
- For example, if $R = pq$ for two large prime numbers p, q, we can construct H_1 with diagonal entries $(R - pq)^2$ for $p, q = 1, \dots, \sqrt{R}$.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.
- For $H_t = (1-t)H_0 + tH_1$ for $t \in [0,1]$, let $|\psi_t\rangle$ be the ground state of H_t, i.e., the eigenvector corresponding to the smallest eigenvalue.
- Under suitable condition, the ground state $(1,\dots,1)^t/\sqrt{2^n}$ of H_0 will change continuously to the ground state $|e_j\rangle$ of H_1.
  We can then determine the smallest diagonal entry of H_1.
- For example, if $R = pq$ for two large prime numbers p, q, we can construct H_1 with diagonal entries $(R - pq)^2$ for $p, q = 1, \dots, \sqrt{R}$.
- How to set up a quantum system with H_1 as the Hamiltonian is challenging.

Adiabatic Quantum Computing Model

- Encode the solution of a problem as the smallest diagonal entry of a large Hermitian matrix $H_1 \in M_N$ with $N = 2^n$.
- Find a Hermitian matrix $H_0 \in M_N$ such as $H_0 = -\sum_{j=0}^{N-1} X_j$, where X_j is a tensor product with $n-1$ copy of I_2 and 1 copy of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at the jth component. For example, if $n = 3$, then $X_0 = XII, X_1 = IXI, X_2 = IIX$.
- For $H_t = (1-t)H_0 + tH_1$ for $t \in [0,1]$, let $|\psi_t\rangle$ be the ground state of H_t, i.e., the eigenvector corresponding to the smallest eigenvalue.
- Under suitable condition, the ground state $(1,\dots,1)^t/\sqrt{2^n}$ of H_0 will change continuously to the ground state $|e_j\rangle$ of H_1.
  We can then determine the smallest diagonal entry of H_1.
- For example, if $R = pq$ for two large prime numbers p, q, we can construct H_1 with diagonal entries $(R - pq)^2$ for $p, q = 1, \dots, \sqrt{R}$.
- How to set up a quantum system with H_1 as the Hamiltonian is challenging.
- There are many other implementation issues.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \binom{a}{b}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.
- We can apply rotation to $|\psi\rangle$ by a unitary matrix U and measure $U|\psi\rangle$ to get additional information.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.
- We can apply rotation to $|\psi\rangle$ by a unitary matrix U and measure $U|\psi\rangle$ to get additional information.
- In quantum physics, $|\psi\rangle$ and $e^{it}|\psi\rangle$ represent the same quantum states.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.
- We can apply rotation to $|\psi\rangle$ by a unitary matrix U and measure $U|\psi\rangle$ to get additional information.
- In quantum physics, $|\psi\rangle$ and $e^{it}|\psi\rangle$ represent the same quantum states.
- So, we may assume that $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $a \geq 0$.

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \binom{a}{b}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.
- We can apply rotation to $|\psi\rangle$ by a unitary matrix U and measure $U|\psi\rangle$ to get additional information.
- In quantum physics, $|\psi\rangle$ and $e^{it}|\psi\rangle$ represent the same quantum states.
- So, we may assume that $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $a \geq 0$.
- Let $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$

Quantum state tomography

- If there is an ensemble of identical copies of a quantum state $|\psi\rangle = a|0\rangle + b|1\rangle = \binom{a}{b}$, then we can do measurement to determine/estimate $|a|^2, |b|^2$.
- But we will not be ale to estimate the actual complex numbers a, b.
- We can apply rotation to $|\psi\rangle$ by a unitary matrix U and measure $U|\psi\rangle$ to get additional information.
- In quantum physics, $|\psi\rangle$ and $e^{it}|\psi\rangle$ represent the same quantum states.
- So, we may assume that $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $a \geq 0$.
- Let $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad G = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$
- Then we can use the measurements of

$$I|\psi\rangle = \binom{a}{b}, \quad H|\psi\rangle = \binom{a+b}{a-b}, \quad G|\psi\rangle = \binom{a+ib}{ia+b}$$

to determine/estimate a, b.

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.
- However, if $|\psi\rangle = (a_1, 0, 0, a_4)$, or $(0, a_2, a_3, 0)^t$, we need to use non-local operation such as the CNOT gate:

$$C = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix} \text{ with } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.
- However, if $|\psi\rangle = (a_1, 0, 0, a_4)$, or $(0, a_2, a_3, 0)^t$, we need to use non-local operation such as the CNOT gate:

$$C = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix} \text{ with } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The use of CNOT gate is expensive and attractive to error.

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.
- However, if $|\psi\rangle = (a_1, 0, 0, a_4)$, or $(0, a_2, a_3, 0)^t$, we need to use non-local operation such as the CNOT gate:

$$C = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix} \text{ with } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The use of CNOT gate is expensive and attractive to error.

- In general, for an n-qubit states $|\psi\rangle$, the number of local unitary operators and CNOT gates needed depend on the zero pattern of $|\psi\rangle$.

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.
- However, if $|\psi\rangle = (a_1, 0, 0, a_4)$, or $(0, a_2, a_3, 0)^t$, we need to use non-local operation such as the CNOT gate:

$$C = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix} \text{ with } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The use of CNOT gate is expensive and attractive to error.

- In general, for an n-qubit states $|\psi\rangle$, the number of local unitary operators and CNOT gates needed depend on the zero pattern of $|\psi\rangle$.
- How to minimize the use of CNOT gates is of interest.

Higher dimensions

- For a two qubit state $|\psi\rangle = (a_1, a_2, a_3, a_4)^t$ with $a_1 \geq 0$, in most cases we can determine $|\psi\rangle$ by the measurements of $|\psi\rangle, U_1|\psi\rangle, \ldots, U_4|\psi\rangle$ with

$$U_1 = I \otimes H = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad U_2 = I \otimes G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

$$U_3 = H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \quad U_4 = G \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}.$$

- These are local measurements that can be done separately for the first and second qubit.
- However, if $|\psi\rangle = (a_1, 0, 0, a_4)$, or $(0, a_2, a_3, 0)^t$, we need to use non-local operation such as the CNOT gate:

$$C = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix} \text{ with } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The use of CNOT gate is expensive and attractive to error.

- In general, for an n-qubit states $|\psi\rangle$, the number of local unitary operators and CNOT gates needed depend on the zero pattern of $|\psi\rangle$.
- How to minimize the use of CNOT gates is of interest.

This is an on-going project with Shuhong Wang, Kevin Wu and Zherui Zhang.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.
- In quantum computing, $N = 2^n$ is an n-qubit system. We can use unitary operators of the form $V_1 \otimes \cdots \otimes V_n$ with $V_j \in \{I, H, G\}$ to determine ρ.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.
- In quantum computing, $N = 2^n$ is an n-qubit system. We can use unitary operators of the form $V_1 \otimes \cdots \otimes V_n$ with $V_j \in \{I, H, G\}$ to determine ρ.
- In this scheme, we need to do 3^n set of such (local) measurements. The number cannot be reduced.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.
- In quantum computing, $N = 2^n$ is an n-qubit system. We can use unitary operators of the form $V_1 \otimes \cdots \otimes V_n$ with $V_j \in \{I, H, G\}$ to determine ρ.
- In this scheme, we need to do 3^n set of such (local) measurements. The number cannot be reduced.
- If non-local unitary operators are allowed, we only need $N + 1 = 2^n + 1$ sets of measurements.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.
- In quantum computing, $N = 2^n$ is an n-qubit system. We can use unitary operators of the form $V_1 \otimes \cdots \otimes V_n$ with $V_j \in \{I, H, G\}$ to determine ρ.
- In this scheme, we need to do 3^n set of such (local) measurements. The number cannot be reduced.
- If non-local unitary operators are allowed, we only need $N + 1 = 2^n + 1$ sets of measurements.
- We "extend" an n-qubit state ρ as a $2n$-qubit state by an operation of the form $\tilde{\rho} = X \rho X^* \in M_{N^2}$, where X is $N \times N^2$ such that $X^* X = I_N$.

Mixed states (Li,Nakahara,Pelejo,Stanish,Wang)

- A general quantum (mixed) state is an $N \times N$ density matrix ρ, A is Hermitian with nonnegative eigenvalues summing up to 1.
- One can use $U_0 = I, U_1, \ldots, U_N$ do measurements of the diagonal entries of $U_j \rho U_j^*$ to determine ρ.
- In quantum computing, $N = 2^n$ is an n-qubit system. We can use unitary operators of the form $V_1 \otimes \cdots \otimes V_n$ with $V_j \in \{I, H, G\}$ to determine ρ.
- In this scheme, we need to do 3^n set of such (local) measurements. The number cannot be reduced.
- If non-local unitary operators are allowed, we only need $N + 1 = 2^n + 1$ sets of measurements.
- We "extend" an n-qubit state ρ as a $2n$-qubit state by an operation of the form $\tilde{\rho} = X \rho X^* \in M_{N^2}$, where X is $N \times N^2$ such that $X^* X = I_N$.
- A measurement of $\tilde{\rho}$ can determine ρ.

Other interesting topics

- Quantum system tomography.

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.
- Separability of quantum states.

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.
- Separability of quantum states.
- Distinguishability of quantum channels.

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.
- Separability of quantum states.
- Distinguishability of quantum channels.
- Quantum error correction, Toric code.

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.
- Separability of quantum states.
- Distinguishability of quantum channels.
- Quantum error correction, Toric code.
- Applications of quantum computing techniques to image processing, neural network, AI, ...

Other interesting topics

- Quantum system tomography.
- Quantum states and quantum operations with special properties.
- Separability of quantum states.
- Distinguishability of quantum channels.
- Quantum error correction, Toric code.
- Applications of quantum computing techniques to image processing, neural network, AI, ...

Hope to tell you more next time.

Thank you for your attention!