BASICS AND NOTATION

Graphs $\quad \Gamma = (V(\Gamma), E(\Gamma))$

Conventions: $|V(\Gamma)|$ finite,

Edges are undirected, no loops,

no multiple edges

$\{\alpha, \beta\} \in E(\Gamma) \longleftrightarrow \alpha \sim \beta$

# of neighbors of vertex: degree or valency

$\mathrm{Aut}(\Gamma) := \{$ bijections $V(\Gamma) \to V(\Gamma)$ preserving $E(\Gamma) \}$

$\mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(V(\Gamma))$

NOTATION If $g \in \mathrm{Aut}(\Gamma)$, $\alpha \in V(\Gamma)$,

we write α^g for the image of

α under g.

α^G : orbit of α under G

(all places α can be sent by group G)

G_α : stabilizer of α in G

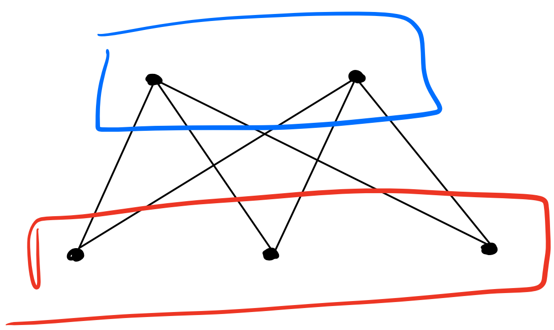
(all elements of G fixing α)

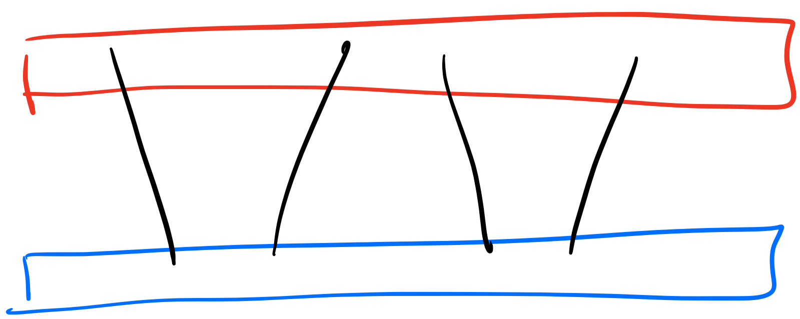
Q: What's a good notion for "a lot" of symmetry?

It would be nice if $Aut(\Gamma)$ is transitive on either vertices or edges

(G is transitive on a set Ω if for any $\alpha \in \Omega$, $\alpha^G = \Omega$)

EX $K_{2,3}$





Nice for incidence structures!

So: maybe ≤ 2 orbits on vertices

FIRST IDEA: "A lot" of "global" symmetry

DEF G is <u>2-transitive</u> on Ω if, for any $\alpha \neq \beta$, $\gamma \neq \delta \in \Omega$, there exists $g \in G$ such that $\underbrace{(\alpha, \beta)}_{\text{ordered pair}}{}^{g} = (\gamma, \delta)$

Which graphs are 2-transitive on $V(\Gamma)$?

Only complete graphs (K_n) (all edges) and empty/null graphs (no edges)!

Pf Pick $\alpha \neq \beta \in V(\Gamma)$. Either $\{\alpha, \beta\} \in E(\Gamma)$ or $\{\alpha, \beta\} \notin E(\Gamma)$.

Then use 2-transitivity. $\square$

BETTER IDEA: Stabilizer G_α is "large"/ has a "robust" action locally (A lot of "local" symmetry)

DEF For $s \in \mathbb{Z}_{\geq 0}$, an <u>s-arc</u> of a graph Γ is a sequence of $s+1$ vertices $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ satisfying:

(1) $\alpha_i \sim \alpha_{i+1}$ for $0 \le i \le s-1$,

(2) $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \le i \le s-1$.

Given $G \le \mathrm{Aut}(\Gamma)$, Γ is <u>(G,s)-arc-transitive</u> if G is transitive on the set of s-arcs in Γ

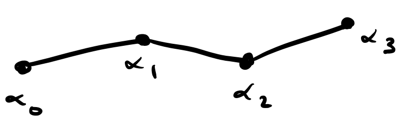
(Γ: <u>s-arc-transitive</u>)

Γ is <u>locally (G,s)-arc-transitive</u> if G_α is transitive on the set of all s-arcs starting @ α ($\alpha_0 = \alpha$) for all $\alpha \in V(\Gamma)$

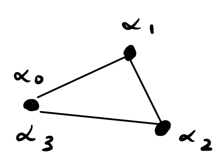
(Γ: <u>locally s-arc-transitive</u>)

In each case, Γ should contain ≥ 1 s-arc

<u>EX</u> K_n (complete graph on n vertices) is 2-arc-transitive but <u>not</u> 3-arc-transitive

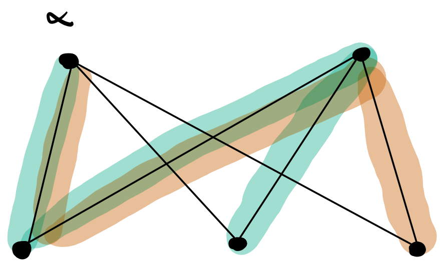


vs.



Complete bipartite graphs $K_{m,n}$ are locally 3-arc-transitive

EX



EX Cycles! C_n is s-arc-transitive for all $s \geq 0$!

THM Let Γ be a graph where all vertices have degree ≥ 3.

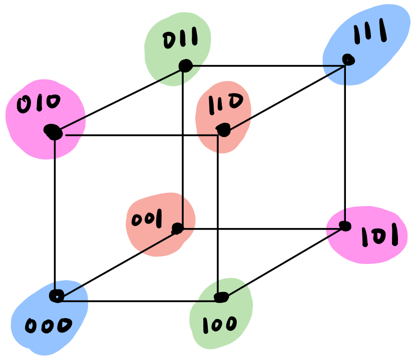
(1) (Tutte 1947, 1959) If Γ is s-arc-transitive and <u>all</u> vertices have degree 3, then $s \leq 5$.

(2) (Weiss 1981) If Γ is s-arc-transitive, then $s \leq 7$.

(3) (van Bon, Stellmacher "2015") If Γ is locally s-arc-transitive, then $s \leq 9$.

Q: What's a good framework for studying these graphs?
(locally s-arc-transitive)

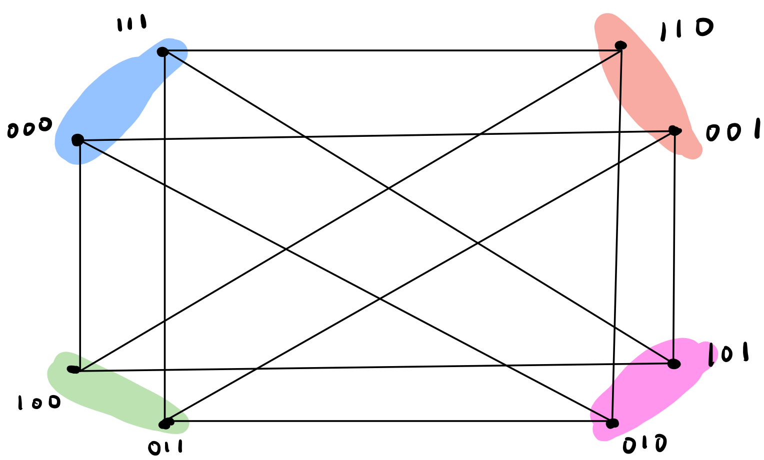
MOTIVATING EX



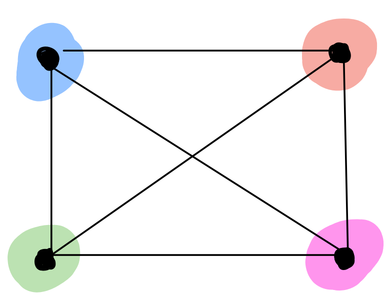
Γ: 2-arc-transitive graph

FACT: $G := \mathrm{Aut}(\Gamma) \cong S_4 \times C_2$

$G_{000} \cong S_3$



$\downarrow$ quotient / "covering projection"



$\bar{\Gamma}$

$\bar{G} = \mathrm{Aut}(\bar{\Gamma})$

Then $\bar{G}_{\overline{000}} \cong S_3$

We maintained the degree of each vertex and the stabilizer!

What do we need from our original graph to do this?

While we need a block system (G-invariant partition of vertices), not just "any" block system "plays nicely" w/ local structure!

KEY: In previous example, G had an intransitive normal subgroup (w/ ≥ 3 orbits)

In such a case, we can create a normal quotient graph Γ_N like above:

$V(\Gamma_N)$: N-orbits,

adjacency: are some two vertices in the orbits adjacent?

THM (1) (Praeger 1993) If Γ is (G,s)-arc-transitive, $s \geq 2$, $N \lhd G$ w/ ≥ 3 orbits on vertices, then Γ_N is connected and $(G/N, s)$-arc-transitive.

(2) (Giudici, Li, Praeger 2004) If Γ is locally (G,s)-arc-transitive, $s \geq 2$, G has <u>two</u> orbits Δ_1, Δ_2 on vertices, $N \lhd G$, intransitive on both orbits (w/ ≥ 3 N-orbits on each G-orbit), then Γ_N is locally $(G/N, s)$-arc-transitive.