GAG Seminar
September 14, 2022

Contact seaweeds

by

Nick Russoniello

A report on joint work
with
Vincent Coll (Lehigh),
Nick Mayers (NC State),
and Gil Salgado (UASLP)

William & Mary

Lie Algebra

$$(\mathfrak{g}, [-,-])$$

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Example

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$A_3 = \mathfrak{sl}(4)$$

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$$[X,Y] = XY - YX$$

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Kirillov Form

$$B_\varphi(x,y) = -d\varphi(x,y) = \varphi([x,y])$$

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Regular

$$\dim(\ker(B_\varphi)) = \text{ind } \mathfrak{g}$$

$$\dim \mathfrak{g} = 2k$$

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$(\mathfrak{g}, \varphi)$ is *Frobenius*

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$$\begin{aligned} &(\mathfrak{g}, \varphi) \text{ is } \textit{Frobenius} \\ &\iff (d\varphi)^k \neq 0 \\ &\iff \operatorname{ind} \mathfrak{g} = 0 \\ &\iff B_\varphi \text{ non-degenerate} \end{aligned}$$

$$\dim \mathfrak{g} = 2k \qquad\qquad \dim \mathfrak{g} = 2k+1$$

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$\implies \ker(B_\varphi) = \text{span}\{h\}$

Useful Equivalence

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$$\mathfrak{g} \text{ contact} \iff \left(\bullet \ \text{regular } \varphi \in \mathfrak{g}^*, \right.$$

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Useful Equivalence

$$\mathfrak{g} \text{ contact} \iff \left(\begin{array}{l} \bullet \ \text{regular } \varphi \in \mathfrak{g}^*, \\[2em] \bullet \ \ker(B_\varphi) = \text{span}\{h\}, \text{ and} \\[2em] \bullet \ \varphi(h) \neq 0. \end{array} \right)$$

Seaweeds

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$$\mathfrak{p} + \mathfrak{p}' = \mathfrak{g} \implies \mathfrak{s} = \mathfrak{p} \cap \mathfrak{p}'$$

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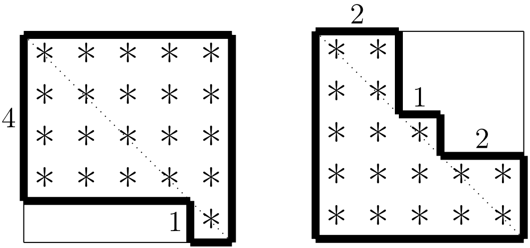

$\mathfrak{p}$ $\mathfrak{p}'$

Seaweeds

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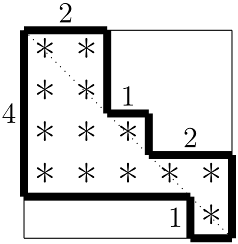

$$\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{s}_5^A \, \frac{4|1}{2|1|2}$$

Example

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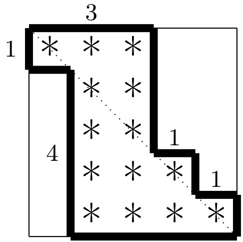

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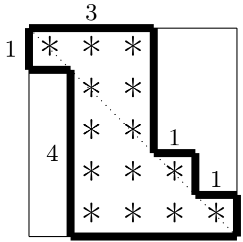

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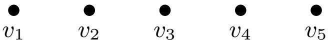

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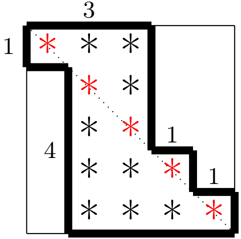

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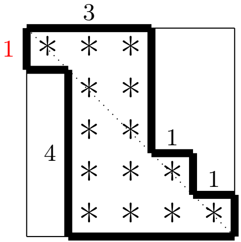

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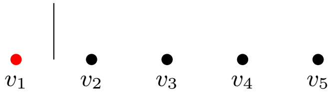

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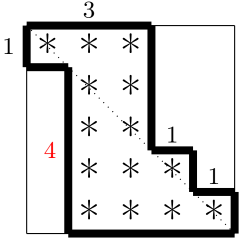

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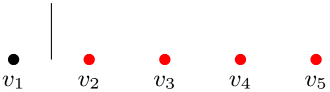

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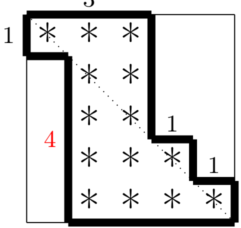

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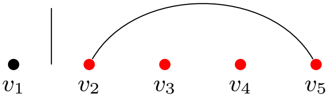

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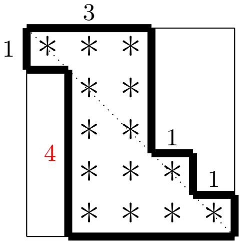

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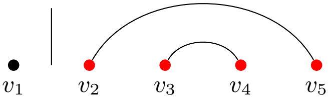

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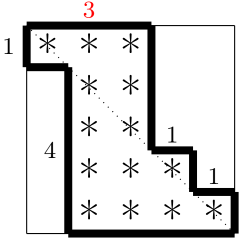

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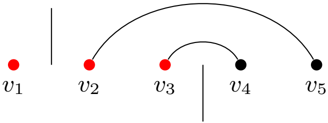

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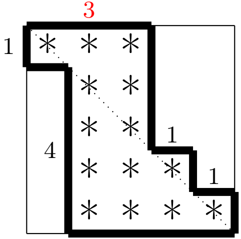

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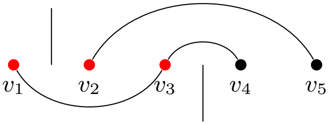

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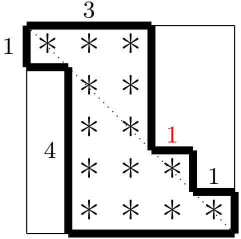

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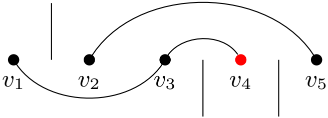

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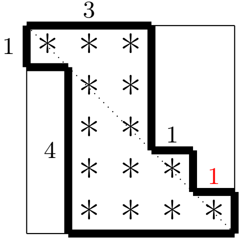

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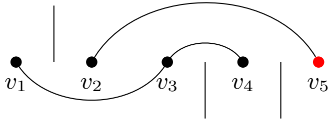

Theorem (Dergachev and Kirillov - J. Lie Theory, 2000)

If $\mathfrak{s}$ is a type-A seaweed, then

$$\operatorname{ind} \mathfrak{s} = 2C + P - 1.$$

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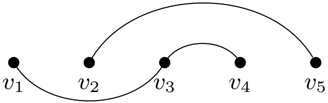

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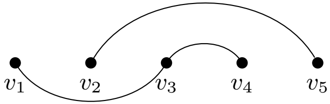

$2(0)$

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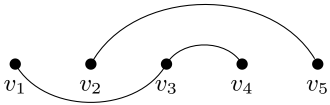

$2(0) + 2$

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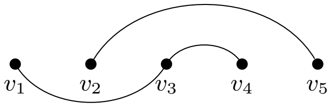

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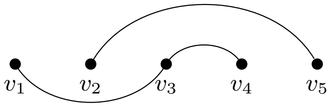

$$2(0) + 2 - 1 = 1$$

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If $\mathfrak{s}$ is a type-A seaweed, then

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Corollary: If $\mathfrak{s}$ is a contact, type-A seaweed, then either $P = 2$ or $C = 1$.

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Wild Conjecture: The "homotopy type" of a seaweed completely determines when it is contact.

Homotopy Type

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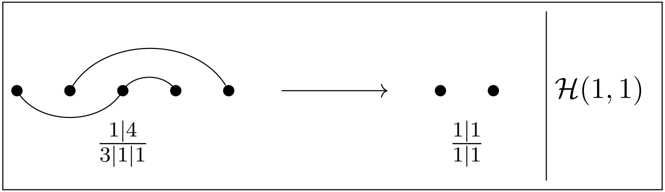
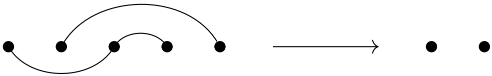
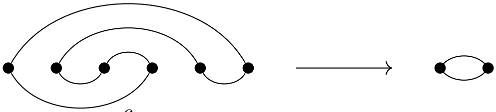


Homotopy Type



$$\frac{1|4}{3|1|1} \longrightarrow \frac{1|1}{1|1} \qquad \mathcal{H}(1,1)$$



$$\frac{6}{4|2} \longrightarrow \frac{2}{2} \qquad \mathcal{H}(2)$$

Theorem (Coll, Mayers, R., and Salgado - Pac. J. Math., 2022)

All index-one, type-A seaweeds are contact.

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Two cases to consider

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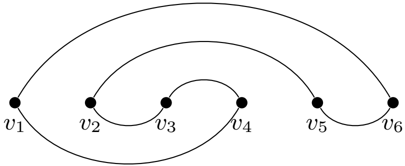
Two cases to consider

$\mathcal{H}(1,1)$ and $\mathcal{H}(2)$

Step 1: Regular φ



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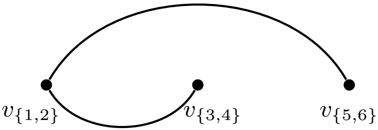

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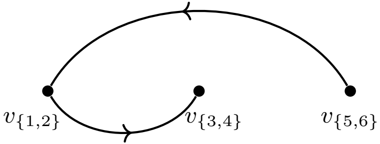

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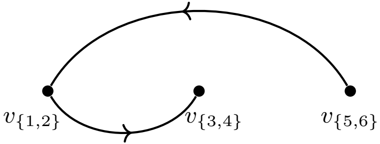

$$\mathfrak{C} = \Big\{ \{1,2\} \times \{1,2\}, \{3,4\} \times \{3,4\}, \{5,6\} \times \{5,6\} \Big\}$$

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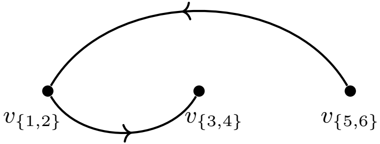

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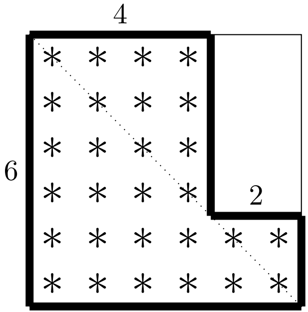

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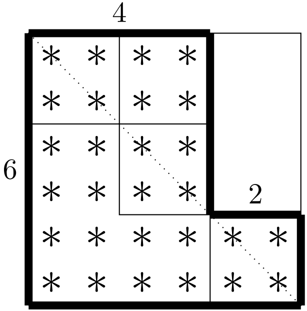

Step 1: Regular φ

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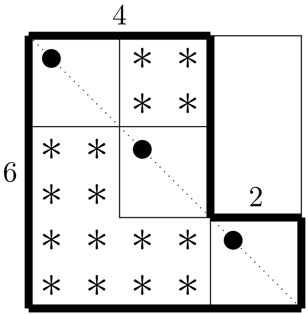

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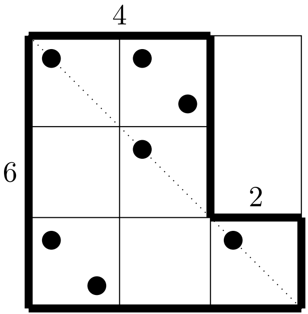

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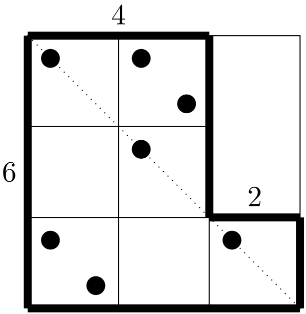

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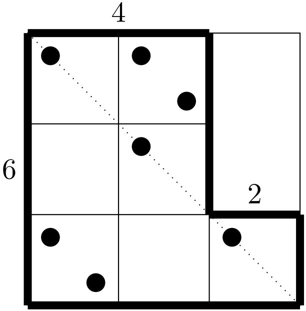

$$\varphi_{(2)} = e^*_{1,1} + e^*_{3,3} + e^*_{5,5}$$
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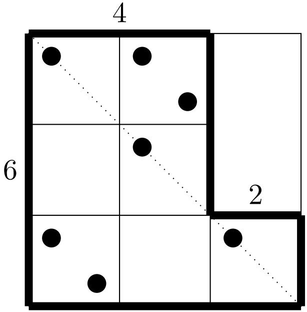
Step 2: $\ker(\mathbf{B}_\varphi) = \mathrm{span}\{\mathbf{h}\}$



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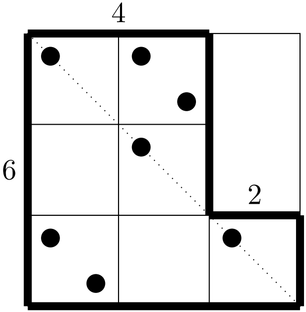

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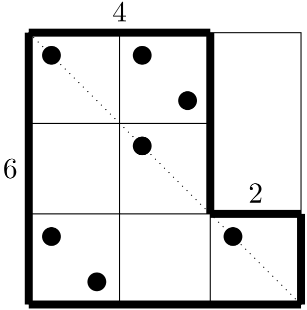

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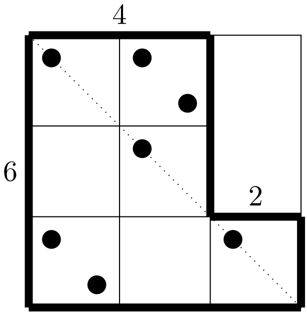

$$\varphi_{(2)} = e_{1,1}^* + e_{3,3}^* + e_{5,5}^*$$
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$$\varphi_{(2)}(h) = 3 \neq 0$$

Let's do the same thing in type C...

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Example

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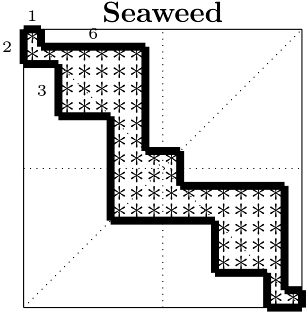

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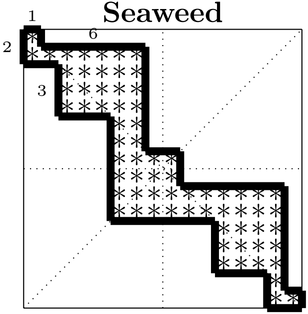

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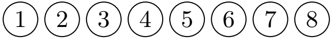

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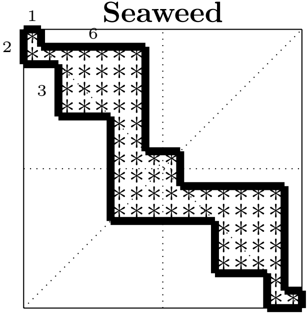

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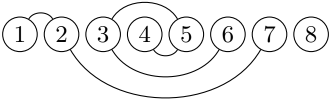

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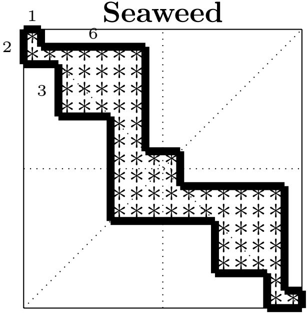

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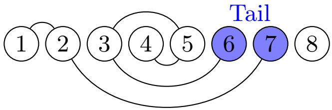

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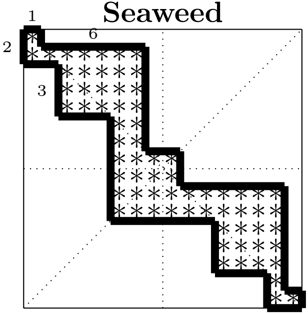

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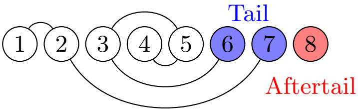

Theorem (Coll, Hyatt, and Magnant - Comm. Algebra, 2017)

If $\mathfrak{s}$ is a type-C seaweed, then

$$\text{ind}\,\mathfrak{s} = 2C + \tilde{P}.$$

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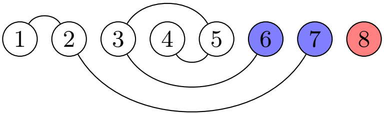

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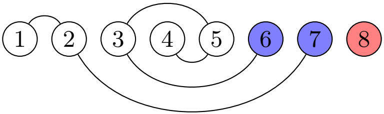

$2(0)$

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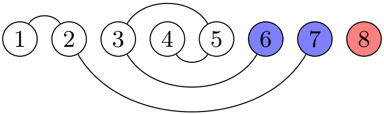

$$2(0) + 1$$

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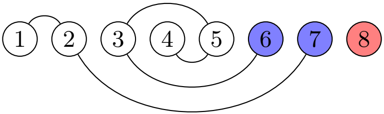
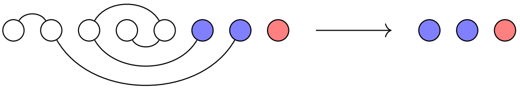


$$2(0) + 1 = 1$$

Homotopy Type

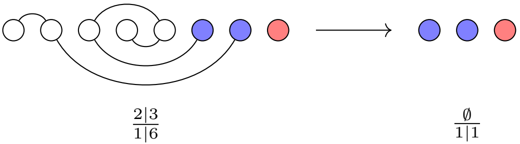
Homotopy Type



$\frac{2|3}{1|6}$ $\frac{\emptyset}{1|1}$

$\mathcal{H}_C(1, 1, 2)$

Homotopy Type



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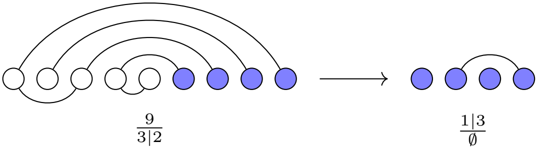

$\mathcal{H}_C(1, 3)$

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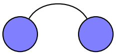

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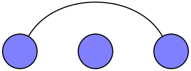

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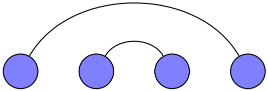

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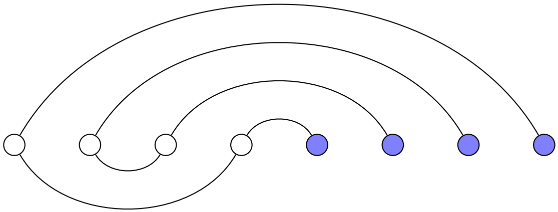

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Theorem (Coll and R. - in preparation)

All index-one, type-C seaweeds are contact.

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Question

Are all index-one seaweeds contact?

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Answer

NO!

Example

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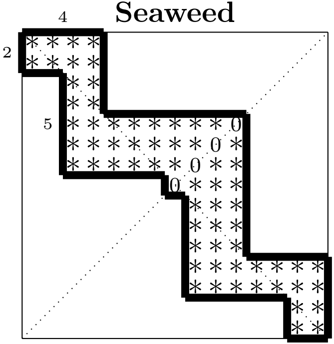

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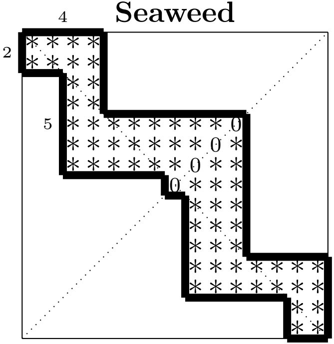

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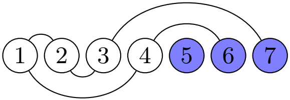

Theorem (Panyushev - Ann. Inst. Fourier, 2005)

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Useful Equivalence

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$\mathfrak{s}$ quasi-reductive $\Longleftrightarrow$

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Theorem (Coll and R. - in preparation)

An index-one seaweed is contact if and only if it is quasi-reductive.

Proof

$$(QR \implies contact)$$

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Choose $\varphi \in \mathfrak{s}^*$ such that $\ker(B_\varphi) = \mathrm{span}\{h\}$, $h \in \mathfrak{s}$ semisimple.

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Generate Cartan-Weyl basis from h;

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If $\varphi(h) = 0$, then define $\varphi' = \varphi + \varepsilon h^*$.

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Therefore, φ' is contact on $\mathfrak{s}$.

$\square$

Theorem (Ammari - J. Lie Theory, 2022)

A seaweed is quasi-reductive if and only if it is "stable."

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A seaweed is quasi-reductive if and only if it is "stable."

Definition

$\varphi \in \mathfrak{g}^*$ is *stable* if there exists $V \ni \varphi$ for which

$$\psi \in V \implies \ker(B_\varphi) \text{ and } \ker(B_\psi) \text{ are conjugate.}$$

Useful Equivalence

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Lemma (Ammari - J. Lie Theory, 2013)

$$\mathfrak{s} \text{ stable} \iff [\ker(B_\varphi), \mathfrak{s}] \cap \ker(B_\varphi) = \{0\}.$$

Proof:

$$(contact \implies stable)$$

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$$\implies [\ker(B_\varphi), \mathfrak{s}] \cap \ker(B_\varphi) = \{0\}$$

$$\iff \mathfrak{s} \text{ stable.}$$

$\square$

$$
\begin{tikzcd}
\text{QR} \arrow[rr, Leftrightarrow, "\text{(Ammari)}"] \arrow[dr, Rightarrow, "\text{(Coll and R.)}"'] & & \text{STABLE} \\
& \text{CONTACT} \arrow[ur, Rightarrow, "\text{(Coll and R.)}"'] &
\end{tikzcd}
$$

Quasi-reductive seaweeds

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- Type A and Type C – Panyushev, 2005

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Seaweeds without seaweed shape

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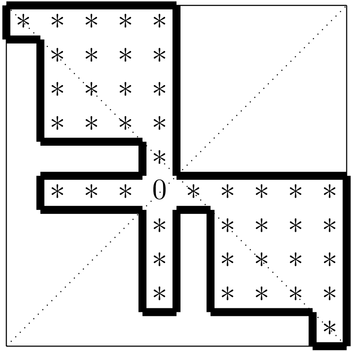

Quasi-reductive seaweeds

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- Type A and Type C – Panyushev, 2005
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- Exceptional – Ammari, 2022
- Type B and Type D parabolics – Duflo, Khalgui, and Torasso, 2012

Facts

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Quasi-reductivity is preserved by winding moves

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Homotopy type identifies the underlying parabolic

Contact Homotopy Types

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- $\mathcal{H}_B(1, 1, \ldots, 1)$

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- $\mathcal{H}_B(1, \ldots, 1, 2)$
- $\mathcal{H}_B(1, \ldots, 1, 3, 1, \ldots, 1)$
- $\mathcal{H}_B(\underbrace{1, \ldots, 1}_{2m}, 2, 1, \ldots, 1)$

Example

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Is $\mathfrak{s}_{17}^{B}\frac{2|3|4|8}{3|10}$ contact?

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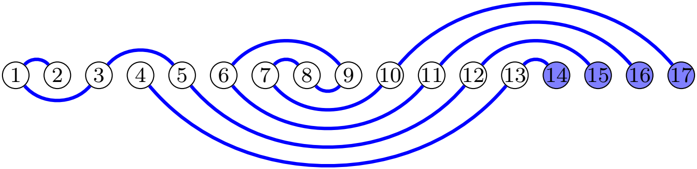

Example

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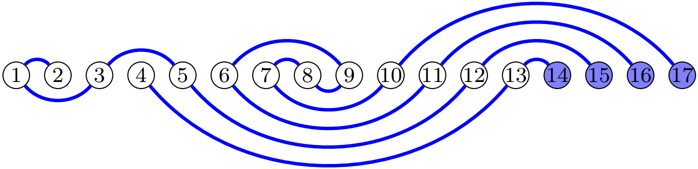

$$\mathcal{H}_B({\color{blue}1, 1, 2}) \implies \text{contact}$$

Thank You