Skateboard Tricks and Topological Flips

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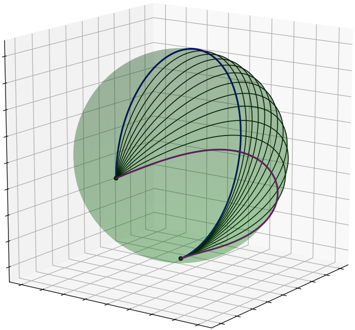


GAG Seminar - October 6th, 2021

Overview

- How this project got started
- A few skateboard flip tricks
- Topological flips
- Homotopies
- The Four Flips Theorem
- Main ideas of the proof
  - Fixing the axis of rotation
  - Quaternions and the 3-sphere
  - Deforming a Double Kickflip to an Ollie
- Open questions

How this project got started

Formed a study group about fundamental groups on Fall 2019 after a Math Club talk about the 3-sphere.

Searching for a project for a few of the students to work on.

Reminiscing about skating and playing Tony Hawk on the PlayStation as kid.

Studying the origins of Geometric Mechanics. Arnold's 1966 paper. "Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits"

How this project got started

"In the year 1765, L. Euler published the equations of rigid body motion which bear his name. It does not seem useless to mark the 200th anniversary of Euler's equations by a modern exposition of the question.

The eulerian motions of a rigid body are the geodesics on the group of rotations of three dimensional euclidean space endowed with a left invariant metric." (translated by Alain Chenciner)

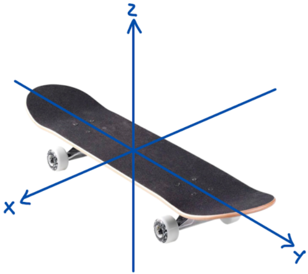
$$\mathbb{I}\dot{\omega} + \omega \times (\mathbb{I}\omega) = 0$$

A few skateboard flip tricks

Let me show you a few YouTube videos of some basic skateboard tricks. We can call this “establishing notation”.

Topological Flips

We now define the resting position for the skateboard in 3-space. The origin of our coordinate system will be placed at the center of mass of the skateboard.



Topological Flips

We then consider an orthonormal frame $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which is locked onto the skateboard and based at its center of mass.

We obtain in this way a time-dependent frame $\vec{v}_1(t), \vec{v}_2(t), \vec{v}_3(t)$. We collect this frame into a 3×3 matrix:

$$R(t) = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{v}_1(t) & \vec{v}_2(t) & \vec{v}_3(t) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Since the frame is orthonormal at every point and continuous in time, the matrix $R(t)$ must satisfy:

$$R(t)^T R(t) = I, \quad \det(R(t)) = 1$$

The matrix $R(t)$ belongs to the special orthogonal group $SO(3)$.

Topological Flips

The initial configuration of the skateboard will always correspond to the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The possible landing configurations are given by the matrix I and the matrix:

$$\mathcal{O} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topological Flips

We define a **topological flip** as a continuous curve:

$$R : [0,1] \to SO(3)$$

such that $R(0) = I$ and $R(1) = I$ or $\mathcal{O}$.

If $\vec{p}$ is a point in the skateboard at the beginning of the trick, then:

$$\vec{p}(t) = R(t)\vec{p} + \vec{d}(t)$$

$R(t)$ describes the rotational motion of the flip trick.

$\vec{d}(t)$ describes the translational motion of the skateboard.

From now on we will ignore the translational motion of the tricks, as it is not relevant for distinguishing between them.

Topological Flips

Most of these topological flips are non-physical in a sense, as they do not satisfy Euler's equations!

Think of them as movements that could be realized if you were controlling the skateboard with your own hands, controlling its movement in the air completely.

Arnold's work shows that the physical flips would correspond to the length-minimizing (geodesic) curves in $SO(3)$, for an appropriate metric constructed from the shape of the skateboard.

A Few Examples

The pop shove-it is given by a left-handed rotation of 180 degrees around the z-axis. It can be described by the following matrix:

$$S(t) = \begin{bmatrix} \cos(\pi t) & \sin(\pi t) & 0 \\ -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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The kickflip is given by a left-handed rotation of 360 degrees around the y-axis. It can be described by the following matrix:

$$K(t) = \begin{bmatrix} \cos(2\pi t) & 0 & -\sin(2\pi t) \\ 0 & 1 & 0 \\ \sin(2\pi t) & 0 & \cos(2\pi t) \end{bmatrix}$$

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We can take advantage of the fact that $SO(3)$ is a topological group in order to create new tricks from old ones.

A Few Examples

The frontside pop shove-it is given by a right-handed rotation of 180 degrees around the z-axis. It can be described by the matrix:

$$S(t)^{-1} = \begin{bmatrix} \cos(-\pi t) & -\sin(-\pi t) & 0 \\ \sin(-\pi t) & \cos(-\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

The heelflip is given by a right-handed rotation of 360 degrees around the y-axis. It can be described by the following matrix:

$$K(t)^{-1} = \begin{bmatrix} \cos(-2\pi t) & 0 & -\sin(-2\pi t) \\ 0 & 1 & 0 \\ \sin(-2\pi t) & 0 & \cos(-2\pi t) \end{bmatrix}$$

A Few Examples

The double kickflip is a left-handed rotation of 720 degrees around the y-axis. It can be described by the following matrix:

$$K(t)^2$$

The 360 shove-it is a left-handed rotation of 360 degrees around the z-axis. It can be described by the following matrix:

$$S(t)^2$$

The 540 shove-it is a left-handed rotation of 540 degrees around the z-axis. It can be described by the following matrix:

$$S(t)^3$$

A Few Examples

The varial kickflip is given by a simultaneous kickflip and pop shove it. It can be described by the following matrix:

$$S(t)K(t)$$

The 360 flip is given by a simultaneous kickflip and 360 pop shove it. It can be described by the following matrix:

$$S(t)^2K(t)$$

The hardflip can be described by the matrix:

$$U(t)K(t/2)$$

Reality is not as simple as we would like

The motion of a varial kickflip is described as a simultaneous backside 180 shove-it and kickflip. We may represent this trick by $S(t)K(t)$. However, this representation of a varial kickflip has a number of issues.

The notion of a "simultaneous motion of a kickflip and backside shove-it" is not well defined, this is because the multiplication operation in $SO(3)$ is not commutative. For example the curve $K(t)S(t)$ can also be described as a simultaneous kickflip and backside shove-it, however its movement does not resemble anything that a skater might call a varial kickflip.

Reality is not as simple as we would like

Furthermore, the topological flip $S(t)K(t)$ is not physical in the sense that it does not satisfy Euler's equation of rigid body motion (and in fact this is true no matter the choice of moments of inertia for the skateboard).

However the description of a simultaneous backside 180 shove-it and kickflip makes sense at the level of Lie algebras. This is related to our current work on this problem.

Homotopies

We are now able to discuss when can two topological flips $R(t)$ and $F(t)$ be continuously deformed into each other.

Recall that a homotopy between $R(t)$ and $F(t)$ is a continuous function $H : [0,1] \times [0,1] \to SO(3)$ such that:

i) $H(0,t) = R(t)$ for all $t \in [0,1]$

ii) $H(1,t) = F(t)$ for all $t \in [0,1]$

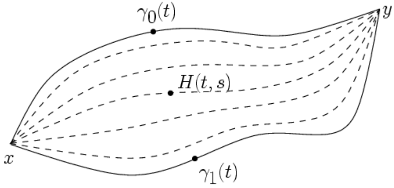
iii) $H(s,0) = I_3$ for all $s \in [0,1]$

iv) $H(s,1) = I_3$ or $\mathcal{O}$ for all $s \in [0,1]$

Notice that for such a homotopy between $R(t)$ and $F(t)$ to exist it must be the case that $R(1) = F(1)$.

Homotopies

We will say that these tricks are *homotopic* and we will denote $R(t) \approx F(t)$.



Source: Wikimedia Commons, created by Yonatan

The relation $\approx$ defines an equivalence relation on the set of topological flips. We call an equivalence class of this relation a *homotopy class* and we denote the homotopy class of a given topological flip $R(t)$ by $[R(t)]$.

The Four Flips Theorem

Theorem ('20 CHHM)

There are exactly four homotopy classes of topological flips. The set of homotopy classes has a natural group structure isomorphic to the cyclic group $\mathbb{Z}/4\mathbb{Z}$.

The homotopy class of the pop shove-it $[S(t)]$ is a generator of this group. My preferred choice of representatives is:

$$\begin{array}{ccl} 0 & \leftrightarrow & \text{Ollie} \\ 1 & \leftrightarrow & \text{Shuvit} \\ 2 & \leftrightarrow & \text{Kickflip} \\ 3 & \leftrightarrow & \text{Varial Kickflip} \end{array}$$

Main Ideas in the proof

One is able to "fix the axis of rotation". We may show that any topological topological flip $F(t)$ is homotopic to a curve of the form:

$$F_0(t) = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & 0 \\ \sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And then show that the 720 shove-it is contractible to the constant curve at the identity matrix. Or equivalently, show that the double kickflip is homotopic to the ollie.

Main Ideas in the proof: Quaternions

In order to describe the geometry of $SO(3)$ we use the quaternion numbers $\mathbb{H}$:

$$\omega = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where a, b, c and d are real numbers. We multiply these numbers as complex numbers, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and using the commutation rules $\mathbf{ij} = \mathbf{k}$, $\mathbf{ki} = \mathbf{j}$ and $\mathbf{jk} = \mathbf{i}$.

The modulus is defined by:

$$|\omega| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

it satisfies the identity $|\omega\alpha| = |\omega| \cdot |\alpha|$

Main Ideas in the proof: Unit 3-sphere

We denote by S^3 the set of unit quaternions ($|\omega| = 1$). This set can be identified with the unit sphere in $\mathbb{R}^4$

$$\{(a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

Notice that the quaternionic multiplication defines a group operation on S^3.

Let $I\mathbb{H}$ denote the set of imaginary quaternions $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, this is isomorphic to $\mathbb{R}^3$ as a real vector space.

Main Ideas in the proof: Conjugation action

If $\vec{v}$ is an imaginary quaternion and u is a unit quaternion, then the conjugate $u\vec{v}u^{-1}$ is also an imaginary quaternion. Furthermore:

$$|u\vec{v}u^{-1}| = |u||\vec{v}||u^{-1}| = |\vec{v}|$$

Which defines an action of S^3 by isometries on $\mathbb{R}^3$. We obtain a surjective group homomorphism:

$$\nu : S^3 \to SO(3)$$

and the kernel of ν is $\ker \nu = \{1, -1\}$.

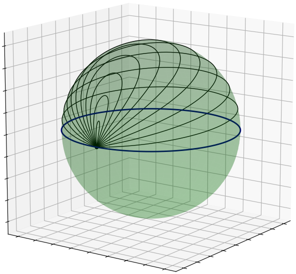
With this homomorphism we may show that $SO(3) \cong \mathbb{R}P^3$

Main Ideas in the proof: Contracting a double-kickflip

The curve $K(t)$ in $SO(3)$ describing a double kickflip, “lifts” to an equator $\kappa(t)$ on S^3.

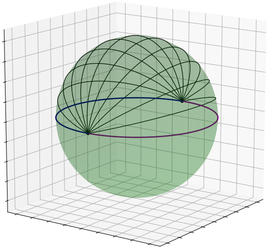
$$\kappa(t) = \cos(\pi t) - \sin(\pi t)\mathbf{j}$$

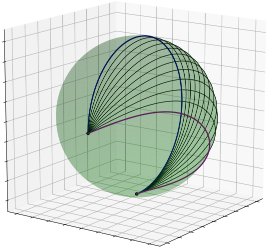
We can then contract the equator to a point!



A few homotopies

Below you can see a homotopy between a kickflip and a heelflip, and a homotopy between a varial kickflip and a 540 shove-it.





Main Ideas in the proof: Coset space

One could prove this theorem in a different way as well. You may notice that a topological flip is in fact a loop in the coset space $SO(3)/\langle\mathcal{O}\rangle$. The group homomorphism $\nu : S^3 \to SO(3)$ yields a homeomorphism:

$$SO(3)/\langle\mathcal{O}\rangle \cong S^3/\langle\mathbf{j}\rangle$$

Since $\langle\mathbf{j}\rangle \cong \mathbb{Z}/4\mathbb{Z}$ and since S^3 is simply-connected we obtain:

$$\pi_1\Big(SO(3)/\langle\mathcal{O}\rangle\Big) \cong \pi_1\Big(S^3/\langle\mathbf{j}\rangle\Big) \cong \langle\mathbf{j}\rangle \cong \mathbb{Z}/4\mathbb{Z}$$

Open questions

- Deformations through physical flips
  - Interesting connections to Lie theory and symplectic geometry
  - Numerical integration
- A geometric characterization of the different flip tricks
- Die rolls.

If you are interested

Our preprint "Skateboard Tricks and Topological Flips" is on arXiv:

arxiv.org/abs/2108.06307

Our Python library is on GitHub:

github.com/holomorpheus/topological-flips

That's all folks!

Thank you!