Harmonic forms on pinched surfaces

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joint work with Peter Buser, Eran Makover and Robert Silhol
'The fox knows many little things, 
*but the hedgehog knows one big thing*'.

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1. Surfaces and geodesics

2. Harmonic vector fields on Riemannian surfaces

3. Riemann surfaces with short simple closed geodesics
   - Riemann surfaces
   - separating case
   - non-separating case
A Riemannian surface is a surface where we can measure angles, distances and area.

- **Note:** The neighborhood of two different points can be different.
- For example, a disk of radius 1, might have a different shape and area at each point.
In the **Euclidean plane** the shortest path between two points is a **straight line**.

- A **shortest path** generalizes the notion of a straight line.
- A **geodesic** is a curve, that is "locally" a shortest path.
- Locally a geodesic arc is like a **rubber band**.
We are interested in compact, orientable surfaces without boundary. Up to deformation these can be classified by their "number of holes".

- The number of holes is the genus $g$ of the surface $S$.
- The surfaces of genus $g \geq 2$ look like glued tori or pretzel surfaces.
A **canonical homology basis** for a surface $S$ of genus $g$ is a set $(\alpha_1, \alpha_2, \ldots, \alpha_{2g})$ of $2g$ simple closed curves, such that

- The curves come in pairs.
- Each pair has exactly one point of intersection.
- The pairs are mutually disjoint.
A 1-form $w$ is a vector field on a Riemannian surface.

A function $f : S \to \mathbb{R}$ can be interpreted as a membrane layed over the surface.

A vector field $w$ is closed if it is locally the gradient of a function $f$, i.e. $w = \text{grad } f$, i.e. if it has a potential function.

In this case the direction of a vector of $w = \text{grad } f$ indicates the direction of the strongest increase of the function $f$.

The length of a vector of $w = \text{grad } f$ indicates the magnitude of the increase.
Harmonic functions with boundary conditions

- We think about the function as a membrane.
- The energy $E(\text{grad } f)$ of a function is given by:
  \[
  E(\text{grad } f) = \lim_{n \to \infty} \sum_{i=1}^{n} \ell(v_i)^2 \cdot \text{area}(U_i).
  \]
- For fixed boundary values we try to find the membrane that has minimal global tension.
- This function $f$ has minimal energy. Such a function is called harmonic.
- In this case the function solves the Dirichlet problem.
Harmonic vector fields on a Riemannian surface

There is no global harmonic function $f$ on a surface $S$ except for the constant function. However, there are harmonic vector fields.

A harmonic vector field can be integrated over a loop of the homology basis.

Take $(\alpha_1, \alpha_2, \ldots, \alpha_{2g})$. A dual basis of harmonic vector fields $(\sigma_1, \sigma_2, \ldots, \sigma_{2g})$ is given by

$$\int_{\alpha_i} \sigma_j = \delta_{ij}, \quad \text{for} \quad i, j \in \{1, 2, \ldots, 2g\}.$$

Example Take $\sigma_2$

$$\int_{\alpha_1} \sigma_2 = 0, \quad \int_{\alpha_2} \sigma_2 = 1, \quad \int_{\alpha_3} \sigma_2 = 0, \ldots$$
We can still get a harmonic function if we cut the surface open. However, the exact boundary values are unknown. Only the difference between boundary values on both sides is known. A harmonic form has minimal energy among all forms with the same periods.

**Example** Take $\sigma_2$ with antiderivative $F_2$

$$\int_{\alpha_2} \sigma_2 = 1 \iff F_2(p_2) - F_2(p_1) = 1$$

**Note:** The harmonic vector field is the vector field with the minimal energy under the given integral conditions.
The **hyperbolic plane** $\mathbb{H}$ is an open disk with radius 1. **Geodesics** are straight lines through the center or half-circles meeting the boundary at an angle of 90 degrees.
Definition: A Riemann surface $S$ of genus $g \geq 2$ is surface of constant curvature $-1$. It can be obtained by gluing a hyperbolic polygon with $4g$ sides by gluing opposite sides.
Collar lemma A short curve $\gamma$ in Riemann surfaces has a large collar $C(\gamma)$. $C(\gamma)$ can be mapped conformally onto a thin flat cylinder $C$. 
Conformal maps

A **conformal map** $\phi : S_1 \rightarrow S_2$ is a map that preserves angles. Conformal maps also preserve the energy.
Short separating simple closed geodesics
Short separating simple closed geodesics

Idea: If the "constraint" is on one side the harmonic vector field vanishes on the other side.

Theorem (Vanishing theorem)

Let $S$ be a Riemann surface of genus $g \geq 2$ and $\gamma$ be separating, such that $\ell(\gamma) \leq \frac{1}{2}$. Let $\sigma$ be a real harmonic vector field, such that $\int_{\alpha_i} \sigma = 0$ for all $\alpha_i \subset S_2$. Then

$$E_{S_2}(\sigma) \leq 2 \cdot 10^4 \cdot \exp \left( -\frac{2 \cdot \pi^2}{\ell(\gamma)} \right) \cdot E_S(\sigma).$$
Short non-separating simple closed geodesics

Theorem (Non-separating case)

Let $\gamma = \alpha_1$ be non-separating, such that $\ell(\gamma) \leq \frac{1}{2}$. For the energies $E(\sigma_1)$ and $E(\sigma_2)$ of the canonical harmonic vector fields $\sigma_1$ and $\sigma_2$ we obtain

$$E(\sigma_1) \text{ is of order } \frac{1}{\ell(\alpha_1)} \quad \text{and} \quad E(\sigma_2) \text{ is of order } \ell(\alpha_1).$$
Local conclusion

- Harmonic vector fields on Riemannian surface can be understood intuitively via their energy minimizing property.
- Harmonic vector fields can be well approximated in collars.
- **Outlook:** We can use this fact to get insight into the Uniformization of surfaces.
- Collars are as important as disks in Riemannian geometry.
Global conclusion

- Harmonic vector fields can be easily understood, but they are hard to master.
- Ideals and dreams are easy to have but hard to realize.
Thank you for your attention!