Harmonic forms on pinched surfaces

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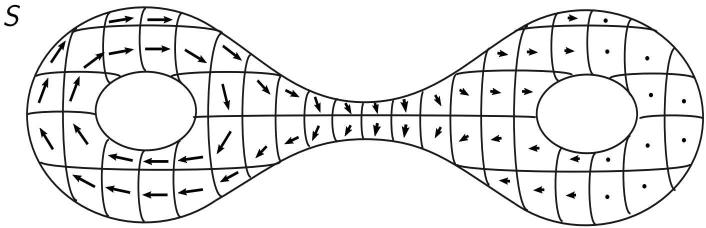

joint work with Peter Buser, Eran Makover and Robert Silhol

Harmonic forms on pinched surfaces

'The fox knows many little things,
but the hedgehog knows one big thing'.
Archilochus (680 - 645 BC)

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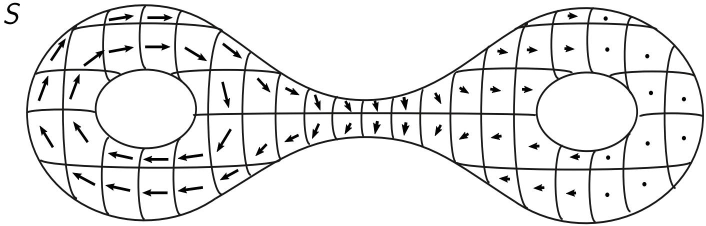

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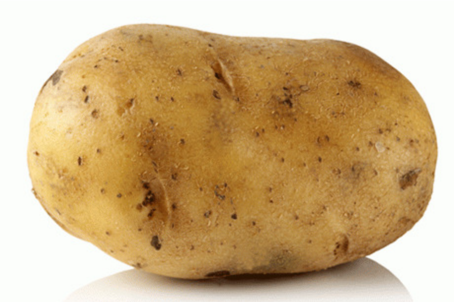
Overview

1. Surfaces and geodesics
2. Harmonic vector fields on Riemannian surfaces
3. Riemann surfaces with short simple closed geodesics
   - Riemann surfaces
   - separating case
   - non-separating case

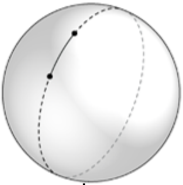
Riemannian surfaces



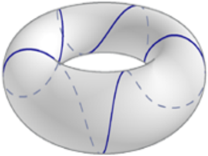
A **Riemannian surface** is a **surface** where we can **measure angles**, **distances** and **area**.

- **Note:** The neighborhood of two different points can be different.
- For example, a disk of radius 1, might have a different shape and area at each point.

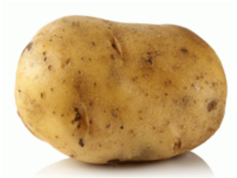
Geodesics on surfaces



sphere



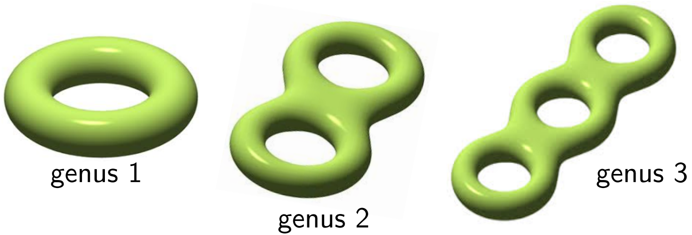
torus



potato

- In the **Euclidean plane** the shortest path between two points is a **straight line**.
- A **shortest path** generalizes the notion of a straight line.
- A **geodesic** is a curve, that is "locally" a shortest path.
- Locally a geodesic arc is like a **rubber band**.

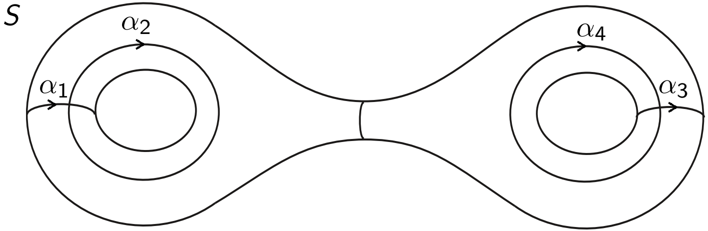
Topology of compact surfaces without boundary



We are interested in **compact**, orientable **surfaces without boundary**. Up to deformation these can be classified by their "number of **holes**".

- The number of holes is the **genus** g of the surface S.
- The surfaces of genus $g \geq 2$ look like glued tori or pretzel surfaces.

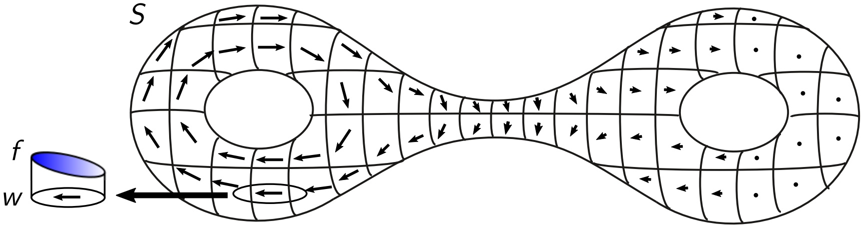
Canonical homology basis



A **canonical homology basis** for a surface S of **genus** g is a set $(\alpha_1, \alpha_2, \ldots, \alpha_{2g})$ of $2g$ simple closed curves, such that

- The curves come in pairs.
- Each pair has exactly one point of intersection
- The pairs are mutually disjoint.

Closed vector fields on a Riemannian surface

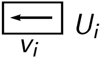


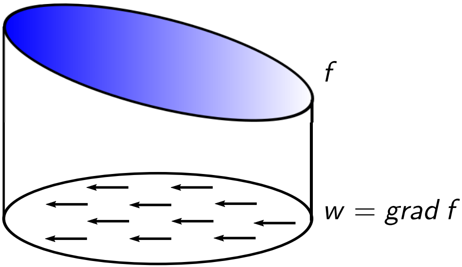
- A **1-form** w is a **vector field** on a **Riemannian surface**.
- A **function** $f : S \to \mathbb{R}$ can be interpreted as a **membrane** layed over the surface.
- A **vector field** w is **closed** if it is locally the gradient of a function f i.e. $\boxed{w = grad\, f}$, i.e. if it has a potential function.
- In this case the **direction of a vector** of $w = grad\, f$ indicates the **direction of the strongest increase** of the function f.
- The **length of a vector** of $w = grad\, f$ indicates the **magnitude of the increase**.

Harmonic functions with boundary conditions

$$E(grad\, f) = \int_S \|grad\, f\|_2^2 \; dA$$

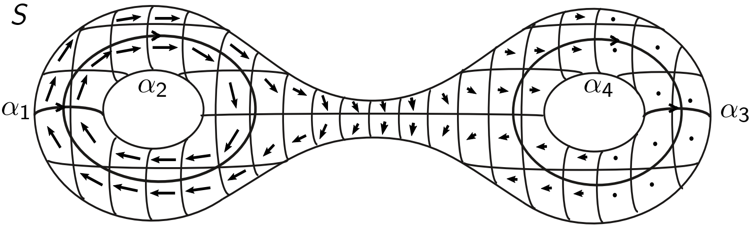
$$E(grad\, f) = \lim_{n\to\infty} \sum_{i=1}^{n} \ell(v_i)^2 \cdot \text{area}(U_i)$$





- We think about the function as a membrane.
- The **energy** $E(grad\, f)$ of a function is given by $\boxed{E(grad\, f) = \lim_{n\to\infty} \sum_{i=1}^{n} \ell(v_i)^2 \cdot \text{area}(U_i)}$.
- For **fixed boundary values** we try to find the membrane that has minimal global tension.
- This function f has **minimal energy**. Such a function is called **harmonic**.
- In this case the function solves the **Dirichlet problem**.

Harmonic vector fields on a Riemannian surface



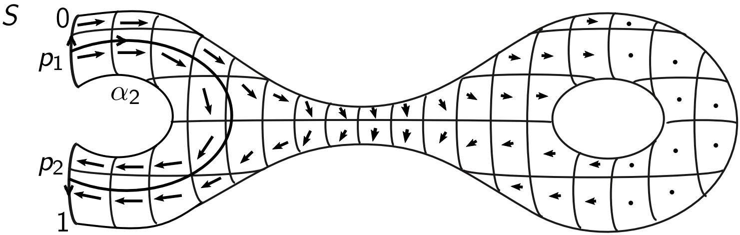
- There is no global **harmonic function** f on a surface S except for the constant function. However, there are harmonic vector fields.
- A **harmonic vector field** can be integrated over a loop of the homology basis.
- Take $(\alpha_1, \alpha_2, \ldots, \alpha_{2g})$. A **dual basis of harmonic vector fields** $(\sigma_1, \sigma_2, \ldots, \sigma_{2g})$ is given by

$$\int_{\alpha_i} \sigma_j = \delta_{ij} \ , \quad \text{for} \quad i, j \in \{1, 2, \ldots, 2g\}.$$

- **Example** Take σ_2

$$\boxed{\int_{\alpha_1} \sigma_2 = 0 \ , \ \int_{\alpha_2} \sigma_2 = 1 \ , \ \int_{\alpha_3} \sigma_2 = 0, \ldots}$$

Harmonic vector fields on a Riemannian surface



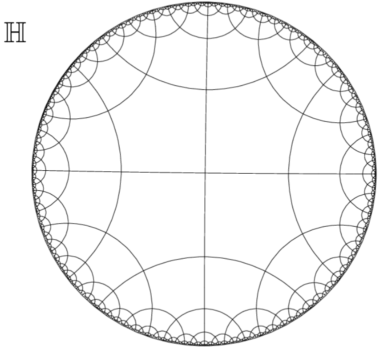
We can still get a harmonic function if we cut the surface open. However the exact boundary values are unknown. Only the difference between boundary values on both sides is known.
A harmonic form has minimal energy among all forms with the same periods.

- **Example** Take σ_2 with antiderivative F_2

$$\boxed{\int_{\alpha_2} \sigma_2 = 1 \Leftrightarrow F_2(p_2) - F_2(p_1) = 1}$$

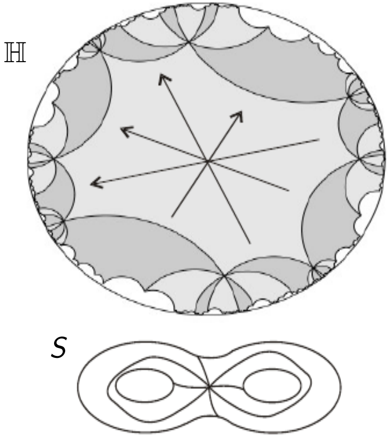
Note: The **harmonic vector field** is the vector field with the **minimal energy** under the given integral conditions.

Hyperbolic plane $\mathbb{H}$



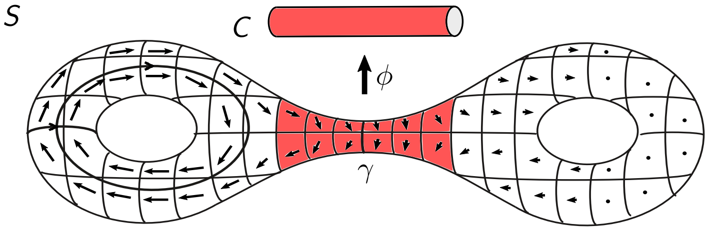
The **hyperbolic plane** $\mathbb{H}$ is an open disk with radius 1. **Geodesics** are straight lines through the center or half-circles meeting the boundary at an angle of 90 degrees.

Riemann surfaces



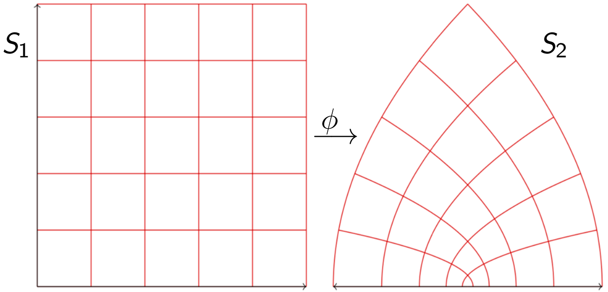
Definition: A **Riemann surface** S of genus $g \geq 2$ is surface of **constant curvature -1**. It can be obtained by gluing a hyperbolic polygon with $4g$ sides by gluing opposite sides.

Short simple closed geodesic



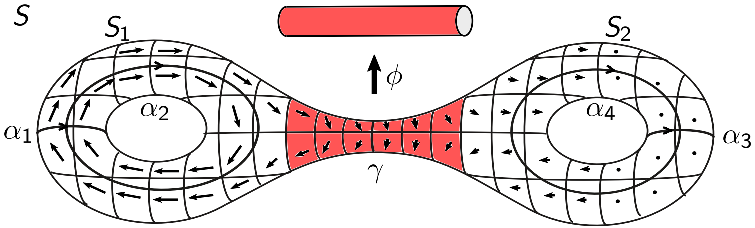
Collar lemma A short curve γ in Riemann surfaces has a large collar $C(\gamma)$.
$C(\gamma)$ can be mapped conformally onto a thin flat cylinder C.

Conformal maps

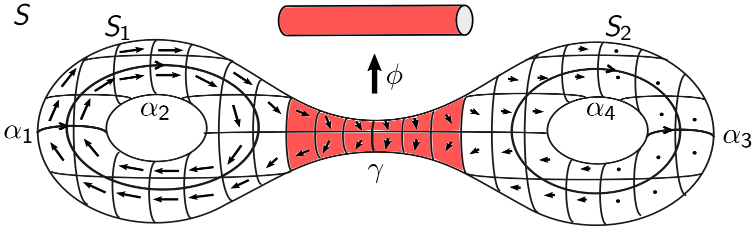


A **conformal map** $\phi : S_1 \to S_2$ is a map that preserves angles. Conformal maps also preserve the energy.

Short separating simple closed geodesics



Short separating simple closed geodesics



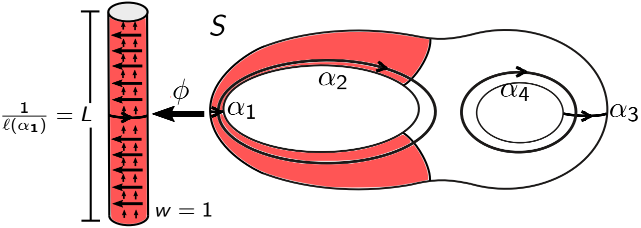
Idea: If the "constraint" is on one side the harmonic vector field vanishes on the other side.

Theorem (Vanishing theorem)

Let S be a Riemann surface of genus $g \geq 2$ and γ be separating, such that $\ell(\gamma) \leq \frac{1}{2}$. Let σ be a real harmonic vector field, such that $\int_{\alpha_i} \sigma = 0$ for all $\alpha_i \subset S_2$. Then

$$E_{S_2}(\sigma) \leq 2 \cdot 10^4 \cdot \exp\left(-\frac{2 \cdot \pi^2}{\ell(\gamma)}\right) \cdot E_S(\sigma).$$

Short non-separating simple closed geodesics



Theorem (Non-separating case)

Let $\gamma = \alpha_1$ be non-separating, such that $\ell(\gamma) \leq \frac{1}{2}$. For the energies $E(\sigma_1)$ and $E(\sigma_2)$ of the canonical harmonic vector fields σ_1 and σ_2 we obtain

$$E(\sigma_1) \text{ is of order } \frac{1}{\ell(\alpha_1)} \quad \text{and} \quad E(\sigma_2) \text{ is of order } \ell(\alpha_1).$$

Local conclusion

- Harmonic vector fields on Riemannian surface can be understood intuitively via their energy minimizing property.
- Harmonic vector fields can be well approximated in collars.
- **Outlook:** We can use this fact to get insight into the Uniformization of surfaces.
- Collars are as important as disks in Riemannian geometry.

Global conclusion

- Harmonic vector fields can be easily understood, but they are hard to master.
- Ideals and dreams are easy to have but hard to realize.

Thank you for your attention!