Toward a Holographic Transform for the Quantum Clebsch-Gordan Formula

Ethan Shelburne

William & Mary

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Representation Theory Basics

Informally, representation theory is about representing elements of an abstract algebraic structure (such as a group) as actions on a vector space.

Definition (Group Representation)

A representation of a group G is a group homomorphism:

$$\pi : G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the general linear group on a vector space V.

Definition (Subrepresentation, Irreducible)

Assume G has a representation π on V. If U is a subspace of V which is invariant under the action of G, we can consider the map:

$$\pi_U : g \mapsto \pi(g)\mid_U$$

which we call a subrepresentation. A representation $\pi : G \to \mathrm{GL}(V)$ is irreducible if it has no proper, nontrivial subrepresentations.

Definition (Intertwiner)

Assume G has representations π_1 on V_1 and π_2 on V_2. A linear map:

$$T : V_1 \to V_2$$

such that, for all $g \in G$, $v \in V$:

$$T(\pi_1(g)v) = \pi_2(g)T(v)$$

is called an intertwiner or equivariant map.

A bijective intertwiner may be called an isomorphism of modules.

Symmetry Breaking Operator

Let G be a group, G' be a subgroup, and:

$$\pi : G \to GL(V)$$

be an irreducible representation.

We may consider the restriction:

$$\pi|_{G'} : G' \to GL(V)$$

which is a representation of G' (but not necessarily irreducible).

Suppose we also have an irreducible representation:

$$\rho : G' \to GL(W)$$

A symmetry breaking operator is a linear map:

$$\psi : V \to W$$

which is an intertwiner for $\pi|_{G'}$ and ρ.

An Elementary Example

Let $\mathbb{C}[x_1, \ldots, x_n]_{(k)}$ be the space of complex polynomials homogeneous of degree $k \in \mathbb{N}$.

Consider any polynomial:

$$p(x) \in \mathbb{C}[x_1, \ldots, x_n]_{(k)} \qquad \text{where} \qquad x = [x_1, \ldots, x_n]$$

We may present $p(x)$ in the following form, based on decreasing powers of x_n:

$$p(x) = \sum_{\ell=0}^{k} r_\ell(x')x_n^{k-\ell} \qquad \text{where} \qquad x' = [x_1, \ldots, x_{n-1}]$$

For any $0 \leq \ell \leq k$, we can thus consider the map:

$$\psi_\ell : \mathbb{C}[x_1, \ldots, x_n]_{(k)} \to \mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)}$$
$$p(x) \mapsto r_\ell(x')$$

We immediately observe that ψ_ℓ is linear and decreases the number of variables in its domain.

Let $G = GL(n, \mathbb{C})$. The general linear group G has an irreducible representation π on $\mathbb{C}[x_1, \ldots, x_n]_{(k)}$:

$$\pi(g)p(x) = p(xg) \qquad \text{and} \qquad x = [x_1, \ldots, x_n]$$

where $g \in G$ and $p(x) \in \mathbb{C}[x_1, \ldots, x_n]_{(k)}$.

We can embed $G' = GL(n-1, \mathbb{C})$ in G as a subgroup including invertible matrices of the form:

$$\begin{bmatrix} * & \ldots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \ldots & * & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

We may then consider $\pi|_{G'}$, under which G' acts on the first $n-1$ variables in x.

Then, we note G' likewise has an irreducible representation ρ on $\mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)}$.

Now:

$$\psi_\ell : \mathbb{C}[x_1, \ldots, x_n]_{(k)} \to \mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)}$$

intertwines $\pi|_{G'}$ and ρ and is therefore a symmetry breaking operator.

Symmetry Breaking Transforms and Holographic Transforms

We also note that we actually defined a *collection* of symmetry breaking operators.

$$\{\psi_\ell : \mathbb{C}[x_1, \ldots, x_n]_{(k)} \to \mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)} \mid 0 \leq \ell \leq k\}$$

Such a collection is called a symmetry breaking transform.

Suppose we had an intertwiner in the opposite direction:

$$\phi_\ell : \mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)} \to \mathbb{C}[x_1, \ldots, x_n]_{(k)}$$

Such a map is called a holographic operator and increases the number of variables in its domain.

A collection of holographic operators such as:

$$\{\phi_\ell : \mathbb{C}[x_1, \ldots, x_{n-1}]_{(\ell)} \to \mathbb{C}[x_1, \ldots, x_n]_{(k)} \mid 0 \leq \ell \leq k\}$$

is called a holographic transform.

Symmetry

What is the reasoning behind the name "symmetry breaking operator?"

Remark

Roughly speaking, a symmetry breaking operator specializes an irreducible representation of G to a component, which is an irreducible representation of G'.

Thus, it "breaks" some of the symmetries from the larger group G.

Tensor Products of Group Representations

Let G be a group and assume V and W are vector spaces carrying irreducible representations of G.

We have that $G \times G$ acts irreducibly on $V \otimes W$.

$V \otimes W$ also carries a representation of G, embedded diagonally in $G \times G$, which is not necessarily irreducible.

Assume G also has an irreducible representation on some vector space U.

Then, we have that any intertwiner:

$$\psi : V \otimes W \to U$$

is a symmetry breaking operator. Likewise, any intertwiner:

$$\phi : U \to V \otimes W$$

is a holographic operator.

For this talk, we are interested in decompositions of tensor products of finite-dimensional representations of the Lie group:

$$\mathrm{SL}(2, \mathbb{C}) = \{X \in GL(2, \mathbb{C}) \mid \det(X) = 1\}$$

The Lie Algebra $\mathfrak{sl}(2,\mathbb{C})$

We will study the representation theory of $\mathrm{SL}(2,\mathbb{C})$ via the representation theory of the Lie algebra:

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in M_2(\mathbb{C}) \mid \mathrm{Tr}(X) = 0\}$$

under the commutator bracket:

$$[X, Y] = XY - YX$$

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ has a basis:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We can directly compute the commutator brackets of each pair of basis vectors:

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y$$

Definition (Highest Weights)

Let V be a $\mathfrak{sl}(2,\mathbb{C})$-module and $\lambda \in \mathbb{C}$. A vector $v \neq 0$ in V is of weight λ if $Hv = \lambda v$. If $Hv = \lambda v$ and $Xv = 0$, then v is a highest weight vector of weight λ.

Proposition

Any non-zero finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module has a highest weight vector.

Classifying Finite Dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules

Theorem

Let V be a finite dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-module. We have:

1. *V is generated by a highest weight vector v of weight λ.*
2. *The scalar λ is an integer equal to $\dim(V) - 1$.*
3. *As an $\mathfrak{sl}(2,\mathbb{C})$-module, V is unique up to isomorphism.*

Conversely, any finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module generated by a highest weight vector is irreducible.

We will henceforth denote a finite dimensional, irreducible $\mathfrak{sl}(2,\mathbb{C})$-module of highest weight n by $V(n)$.

Theorem

Any finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module is completely reducible, i.e the direct sum of irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules.

Theorem (Clebsch-Gordan)

Let $n \geq m$ be non-negative integers. Then there exists an isomorphism of $\mathfrak{sl}(2, \mathbb{C})$-modules:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m)$$

Proof.

Assume for $0 \leq p \leq m$, there exists a highest weight vector w_p of weight $n+m-2p$ in $V(n) \otimes V(m)$. Then, for each p, there exists a nonzero morphism:

$$\phi_p : V(n+m-2p) \to V(n) \otimes V(m)$$

which maps the highest weight vector of $V(n+m-2p)$ to w_p. Since $V(n+m-2p)$ is irreducible, $\ker(\phi_p) = 0$ and ϕ_p is injective. Since $V(n+m-2p)$ are pairwise non-isomorphic,

$$\bigoplus_{0 \leq p \leq m} \phi_p$$

is injective.

To conclude, we have:

$$\dim\left(\bigoplus_{0 \leq p \leq m} V(n+m-2p)\right) = \sum_{p=0}^{m} (n+m-2p+1) = (m+1)(n+1) = \dim(V(n) \otimes V(m))$$

□

Symmetry Breaking Operators and Holographic Operators for the Clebsch-Gordan formula

Given the Clebsch-Gordan formula:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m)$$

We can naturally search for a symmetry breaking transform:

$$\{\psi_p : V(n) \otimes V(m) \to V(n+m-2p) \mid 0 \leq p \leq m\}$$

and a holographic transform:

$$\{\phi_p : V(n+m-2p) \to V(n) \otimes V(m) \mid 0 \leq p \leq m\}$$

Uniqueness of Symmetry Breaking Operators and Holographic Operators

The decomposition in the Clebsch-Gordan formula is multiplicity free. Thus, by Schur's Lemma, we have:

Proposition

Fix $0 \leq p \leq m \leq n$, and let:

$$T, T' : V(n+m-2p) \to V(n) \otimes V(m)$$

be two nonzero intertwiners between modules in the Clebsch-Gordan formula. Then $\lambda T' = T$ must hold for some $\lambda \in \mathbb{C}$.

Explicit $\mathfrak{sl}(2,\mathbb{C})$-modules

Proposition

Let $\mathbb{C}[x]_{\leq n}$ be the vector space of polynomials of degree less than or equal to n. We define a representation of $\mathfrak{sl}(2,\mathbb{C})$ by the actions:

$$Yp(x) = \frac{d}{dx}p(x)$$

$$Xp(x) = \left(nx - x^2\frac{d}{dx}\right)p(x)$$

$$Hp(x) = \left(2x\frac{d}{dx} - n\right)p(x)$$

for any $p(x) \in \mathbb{C}[x]_{\leq n}$. With this action, we have $\mathbb{C}[x]_{\leq n}$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight n.

Proposition

As explicit $\mathfrak{sl}(2,\mathbb{C})$-modules, we have:

$$\mathbb{C}_{\leq n}[x] \otimes \mathbb{C}_{\leq m}[y] \cong \mathbb{C}[x,y]_{n,m}$$

where $\mathbb{C}[x,y]_{n,m}$ denotes the vector space of polynomials with degree in x less than n and degree in y less than m.

Fourier and Poisson Transforms

Theorem (Molchanov, 2015)

The Poisson transform $M_{n,m,p} : \mathbb{C}[x]_{n+m-2p} \to \mathbb{C}[x,y]_{n,m}$ *intertwines these polynomial spaces as* $\mathfrak{sl}(2,\mathbb{C})$*-modules and satisfies:*

$$M_{n,m,p}(f(x)) = \sum_{s=0}^{m-p} \binom{m-p}{s} \frac{(n-p+s)!}{(n-p)!} (y-x)^{m-s} \left(\frac{d}{dx}\right)^{m-p-s} f(x)$$

For some polynomial $f(x,y)$, we set:

$$f^{(a,b)} = \frac{\partial^{a+b} f}{\partial x^a \partial y^b}$$

Theorem (Molchanov, 2015)

The Fourier transform $F_{n,m,p} : \mathbb{C}[x,y]_{n,m} \to \mathbb{C}[x]_{n+m-2p}$ *intertwines these polynomial spaces as* $\mathfrak{sl}(2,\mathbb{C})$*-modules and satisfies:*

$$F_{n,m,p}(f(x,y)) = \frac{(n+m-2p+1)(n-p+1)!}{(n+m-p+1)!} \times \sum_{\alpha=0}^{p} (-1)^{p-\alpha} \binom{n-p+\alpha}{\alpha} \binom{m-\alpha}{p-\alpha} f^{(j-\alpha,\alpha)}(x,x)$$

Infinite Dimensional Case

Remark

Infinite dimensional, irreducible representations of $SL(2,\mathbb{C})$ have a similar decomposition:

$$W(\lambda) \otimes W(\lambda') \cong \bigoplus_{p \in \mathbb{N}} W(\lambda + \lambda' + 2p)$$

In a recent paper (Annales de l'Institut Fourier, 2019), Michael Pevzner and Toshiyuki Kobayashi introduced holographic transforms for these decompositions.

Quantum Analogue to $\mathfrak{sl}(2,\mathbb{C})$?

Remark

There is no widely agreed upon definition of a quantum group. For our purposes, the term roughly describes a (non-cocommutative) algebraic structure which is a deformation of some classical structure associated with a group.

$$\text{Classical Object} \overset{\text{"deformation"}}{\rightsquigarrow} \text{Quantum Object}$$

Lie group		Lie algebra		Enveloping algebra		Quantum Group
$SL(2,\mathbb{C})$	$\rightsquigarrow$	$\mathfrak{sl}(2,\mathbb{C})$	$\rightsquigarrow$	$U(\mathfrak{sl}(2,\mathbb{C}))$	$\overset{\text{"deformation"}}{\rightsquigarrow}$	$U_q(\mathfrak{sl}(2,\mathbb{C}))$

The Enveloping Algebra of $\mathfrak{sl}(2,\mathbb{C})$

We will now focus on the enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$, denoted $U(\mathfrak{sl}(2,\mathbb{C}))$. We note $U(\mathfrak{sl}(2,\mathbb{C}))$ is a cocommutative Hopf algebra.

Proposition (Poincaré-Birkhoff-Witt)

$U(\mathfrak{sl}(2,\mathbb{C}))$ is isomorphic to the associative algebra generated by the three elements X, Y, H with the three relations:

$$[X,Y]=H, \qquad [H,X]=2X, \qquad [H,Y]=-2Y \tag{1}$$

Moreover, a basis for $U(\mathfrak{sl}(2,\mathbb{C}))$ is given by the set $\{X^iY^jH^k\}_{i,j,k\in\mathbb{N}}$.

We can now study the representation theory of three different algebraic structures simultaneously since:

$$\begin{aligned}
&\mathrm{Hom}_{\text{Lie Group}}(\mathrm{SL}(2,\mathbb{C}), GL(V))\\
\cong&\mathrm{Hom}_{\text{Lie Algebra}}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{gl}(V))\\
\cong&\mathrm{Hom}_{\text{Algebra}}(U(\mathfrak{sl}(2,\mathbb{C})), \mathrm{End}(V))
\end{aligned}$$

q-notation

Fix $q \in \mathbb{C}$ which is not a root of unity. For any integer n, we define:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

Set $[0]! = 1$. Then, we may set:

$$[k]! = [1][2]\cdots[k]$$

Likewise, we set:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

for $0 \leq k \leq n$.

Proposition (q-Binomial Theorem)

If x and y are variables subject to the relation:

$$yx = q^2 xy$$

Then, for $n \geq 0$:

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k$$

The quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$

Definition

Fix $q \in \mathbb{C}$, not a root of unity. We define $U_q(\mathfrak{sl}(2,\mathbb{C}))$ as the associative algebra generated by the four variables E, F, K, K^{-1} with the relations:

$$KK^{-1} = K^{-1}K = 1,$$

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F$$

and:

$$[E,F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

The algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is generated by q-deformations of the basis relations defining $U(\mathfrak{sl}(2,\mathbb{C}))$. This structure is an example of what is called a quantum group.

Remark

i. *Like $U(\mathfrak{sl}(2,\mathbb{C}))$, the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a Hopf algebra. Unlike $U(\mathfrak{sl}(2,\mathbb{C}))$, $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is not cocommutative.*
ii. *An alternate formulation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ allows the Hopf algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ to be recovered from the quantum group structure by setting $q = 1$.*

Highest Weight Theory for $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules

Definition

Let $V^{(q)}$ be a $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module and $\lambda \in \mathbb{C}$. A vector $v \neq 0$ in $V^{(q)}$ is of *weight* λ if $Kv = \lambda v$. If $Kv = \lambda v$ and $Ev = 0$, then v is a *highest weight vector* of weight λ.

Proposition

Any non-zero finite dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$*-module has a highest weight vector.*

Classifying Finite Dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules

Theorem

Let $V^{(q)}$ be a finite dimensional irreducible $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module. We have:

1. *$V^{(q)}$ is generated by a highest weight vector v of weight λ.*
2. *The scalar λ is of the form $\lambda = \varepsilon q^n$ where $\varepsilon = \pm 1$ and $n = \dim(V) - 1$.*
3. *As a $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module, $V^{(q)}$ is unique up to isomorphism.*

Conversely, any finite-dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module generated by a highest weight vector is irreducible.

Theorem

Any finite dimensional $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module is completely reducible, i.e. the direct sum of irreducible $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules.

Quantum Clebsch-Gordan

Let $V_n^{(q)}$ denote a $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module with highest weight q^n.

Theorem (Quantum Clebsch-Gordan)

Let $n \geq m$ be nonnegative integers. There exists an isomorphism of $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules:

$$V_n^{(q)} \otimes V_m^{(q)} \cong V_{n+m}^{(q)} \oplus V_{n+m-2}^{(q)} \oplus \cdots \oplus V_{n-m+2}^{(q)} \oplus V_{n-m}^{(q)}$$

Proof.

It suffices to show there exists a highest weight vector $w_p^{(q)}$ of weight q^{n+m-2p} for $0 \leq p \leq m$. $\square$

How to explicitly intertwine the quantum Clebsch-Gordan formula?

Recall, the Poisson Transform takes the form:

$$M_{n,m,p}(f(x)) = \sum_{s=0}^{m-p} \binom{m-p}{s} \frac{(n-p+s)!}{(n-p)!} (y-x)^{m-s} \left(\frac{d}{dx}\right)^{m-p-s} f(x)$$

As of now, we are not aware of a quantum analogue to Molchanov's geometric construction of this holographic transform.

If one could find a "quantum Poisson transform," Molchanov's algebraic method could presumably be adapted to determine a "quantum Fourier transform."

One thing the classical and quantum Clebsch-Gordan formulas have in common is their proofs, which are both dependent on highest weight vectors.

Theorem (Classical Clebsch-Gordan)

Let $n \geq m$ be non-negative integers. Then there exists an isomorphism of $U(\mathfrak{sl}(2,\mathbb{C}))$-modules:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m)$$

Proof.

Assume for $0 \leq p \leq m$, there exists a highest weight vector w_p of weight $n+m-2p$ in $V(n) \otimes V(m)$. Then, for each p, there exists a nonzero morphism:

$$\phi_p : V(n+m-2p) \to V(n) \otimes V(m)$$

which maps the highest weight vector of $V(n+m-2p)$ to w_p. Since $V(n+m-2p)$ is irreducible $\ker(\phi_p) = 0$ and ϕ_p is injective. Since $V(n+m-2p)$ are pairwise non-isomorphic,

$$\bigoplus_{0 \leq p \leq m} \phi_p$$

is injective.

To conclude, we have:

$$\dim\left(\bigoplus_{0 \leq p \leq m} V(n+m-2p)\right) = \sum_{p=0}^{m}(n+m-2p+1) = (m+1)(n+1) = \dim(V(n) \otimes V(m))$$

$\square$

Lemma

Let $V(n)$ and $V(m)$ be $U(\mathfrak{sl}(2,\mathbb{C}))$ modules. Let v be a highest weight vector of $V(n)$ and v' be a highest weight vector of $V(m)$. Denote

$$v_p = \frac{1}{p!} Y^p v \qquad \text{and} \qquad v'_p = \frac{1}{p!} Y^p v'$$

for $p \geq 0$. Then:

$$w_p = \sum_{i=0}^{p} (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!} v_i \otimes v'_{p-i} \tag{2}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight $n+m-2p$.

Algebraic Approach to Constructing Classical CG Holographic Transform

- Consider a map $\phi_{n,m,p}$ such that, if v is the highest weight vector of $V(n+m-2p)$:
$$\phi_{n,m,p} : v \mapsto w_p$$
- Extend the map $\phi_{n,m,p}$ to the basis $\frac{1}{p!}Y^p v$ using *equivariance*.
- Express $\phi_{n,m,p}$ with an explicit choice of modules $(\mathbb{C}[x]_{\leq n+m-2p} \to \mathbb{C}[x,y]_{n,m})$.
- Algebraically manipulate the map $\phi_{n,m,p}$ to determine polynomial coefficients of the image of some basis vector (a monomial in x).

Definition

Let $n \geq m \geq p \geq 0$. We define:

$$\phi_{n,m,p} : \mathbb{C}[x]_{\leq n+m-2p} \to \mathbb{C}[x,y]_{n,m}$$
$$x^\ell \mapsto \alpha \sum_{t=\omega_1}^{\omega_2} \beta_t x^{n-t} y^{p+\ell-n+t}$$

where:

$$\omega_1 = \max(0, n-p-\ell) \qquad \omega_2 = \min(n+m-p-\ell, n)$$

$$\alpha = \frac{m!\ell!}{(n+m-2p)!(m-p)!}$$

$$\beta_t = \sum_{i=\psi_1(t)}^{\psi_2(t)} \gamma_i \chi_{i,t-i}$$

$$\psi_1(t) = \max(0, t+2p+\ell-n-m) \qquad \psi_2(t) = \min(p,t)$$

$$\gamma_i = \frac{(-1)^i}{i!(p-i)!}$$

$$\chi_{i,j} = \binom{k}{j} \frac{(n-i)!(m-p+i)!}{(n-i-j)!(m-p-k+i+j)!}$$

Theorem (ES)

The map $\phi_{n,m,p}$ intertwines $\mathbb{C}[x]_{\leq n+m-2p}$ and $\mathbb{C}[x,y]_{n,m}$ as explicit $U(\mathfrak{sl}(2,\mathbb{C}))$-modules in the Clebsch-Gordan formula.

We know *a priori* that $\phi_{n,m,p}$ must be a constant multiple of Molchanov's Poisson transform.

Theorem (ES)

The Poisson transform $M_{n,m,p}$ and the intertwiner $\phi_{n,m,p}$ satisfy:

$$M_{n,m,p} = \frac{(-1)^p(n+m-2p)!(m-p)!p!}{m!(n-p)!}\phi_{n,m,p} \tag{3}$$

Adapting Approach to Quantum Case: Explicit Modules

Definition

Let I_q be the two-sided ideal in the polynomial algebra $\mathbb{C}[a,b]$ generated by

$$ba - qab$$

We define the quantum plane as the quotient algebra:

$$\mathbb{C}_q[a,b] = \mathbb{C}[a,b]\Big/I_q$$

which is then subject to the relation:

$$ba = qab$$

The quantum plane has automorphisms σ_a and σ_b defined by:

$$\sigma_a(a) = qa, \qquad \sigma_a(b) = b, \qquad \sigma_b(a) = a, \qquad \sigma_b(b) = qb$$

Moreover, we can define q-analogues to partial derivatives by:

$$\frac{\partial_q(a^m b^n)}{\partial a} = [m]a^{m-1}b^n \qquad \text{and} \qquad \frac{\partial_q(a^m b^n)}{\partial b} = [n]a^m b^{n-1}$$

Adapting Approach to Quantum Case: Explicit Modules

Proposition

Let $\mathbb{C}_q[a,b]_{(n)}$ denote the subspace of $\mathbb{C}_q[a,b]$ which contains polynomials of terms which are homogeneous of degree n.
We define a representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $\mathbb{C}_q[a,b]_{(n)}$ by the actions:

$$Ep = a\frac{\partial_q p}{\partial b}, \qquad Fp = \frac{\partial_q p}{\partial a}b$$
$$Kp = (\sigma_a \sigma_b^{-1})(p), \qquad K^{-1}p = \sigma_b \sigma_a^{-1}(p)$$

for any $p(a,b) \in \mathbb{C}_q[a,b]_{(n)}$. We thus have that $\mathbb{C}_q[a,b]_{(n)}$ is an irreducible $U_q(\mathfrak{sl}(2,\mathbb{C}))$-module with highest weight q^n.

Adapting Approach to Quantum Case: Explicit Modules

We use:

$$\mathbb{C}_q[a, b, c, d]_{(n,m)}$$

to denote the space of polynomials in a, b, c, d which are homogeneous of degree n in a, b and homogeneous of degree m in c, d and subject to the relations:

$$ba = qab \qquad \text{and} \qquad dc = qcd$$

Proposition

We have that, as explicit $U_q((2, \mathbb{C}))$-modules:

$$\mathbb{C}_q[a, b]_{(n)} \otimes \mathbb{C}_q[c, d]_{(m)} \cong \mathbb{C}_q[a, b, c, d]_{(n,m)}$$

Adapting Approach to Quantum Case: Highest Weight Vector

Lemma

Let $n \geq m$, let v_0' be a highest weight vector of weight q^n in $V_n^{(q)}$, and let v_0'' be a highest weight vector of weight q^m in $V_m^{(q)}$. Let us define

$$v_p' = \frac{1}{[p]!} F^p v_0' \qquad \text{and} \qquad v_p'' = \frac{1}{[p]!} F^p v_0''$$

for all $p \geq 0$. Then, for $0 \leq p \leq m$:

$$w_p^{(q)} = \sum_{i=0}^{p} (-1)^i q^{-i(m-2p+i+1)} \frac{[m-p+i]![n-i]!}{[m-p]![n]!} v_i' \otimes v_{p-i}'' \tag{4}$$

is a highest weight vector of weight q^{n+m-2p} in $V_n^{(q)} \otimes V_m^{(q)}$.

Algebraic Construction of Quantum Holographic Transform

Definition

Let $n \geq m \geq p \geq 0$. We define:

$$\phi^{(q)}_{n,m,p} : \mathbb{C}_q[a,b]_{(n+m-2p)} \to \mathbb{C}_q[a,b,c,d]_{(n,m)}$$

$$a^\ell b^k \mapsto \mathrm{A} \sum_{t=\omega_1}^{\omega_2} \mathrm{B}_t a^{n-t} b^t c^{m+n-p-\ell-t} d^{-n+p+\ell+t}$$

where: $\omega_1 = \max(0, n-p-\ell) \qquad \omega_2 = \min(n+m-p-\ell, n)$

$$\mathrm{A} = \frac{[\ell]![m]!}{[n+m-2p]![m-p]!}$$

$$\mathrm{B}_t = \sum_{t=\psi_1(t)}^{\psi_2(t)} \Gamma_i \mathrm{X}_{i,t-i}$$

$$\psi_1(t) = \max(0, t+2p+\ell-n-m) \qquad \psi_2(t) = \min(p,t)$$

$$\Gamma_i = q^{-i(m-2p+i+1)} \frac{(-1)^i}{[i]![p-i]!}$$

$$\mathrm{X}_{i,j} = q^{(2i-n+j)(k-j)} \begin{bmatrix} k \\ j \end{bmatrix} \frac{[n-i]![m-p+i]!}{[n-i-j]![m-p+i-k+j]!}$$

Remark

In the classical case, we were able to compare our algebraically constructed holographic transform to Molchanov's Poisson transform.

In the quantum case, however, all we have is our algebraically constructed form.

Conjectural Form for Quantum Holographic Transform

Conjecture

The map $\phi^{(q)}_{n,m,p}$ may be rewritten such that, for any:

$$f(a,b) \in \mathbb{C}_q[a,b]_{(n+m-2p)}$$

we have:

$$\phi^{(q)}_{n,m,p}(f(a,b)) = \sum_{s=0}^{m-p} \Xi c^s (ac - q^{\zeta} bd)^{m-s} b^{-s} \left(\frac{\partial_q^2}{\partial a \partial b}\right)^{m-p-s} \sigma_b^{\Theta}(f(a,b))$$

where Ξ, ζ, and Θ are constants dependent on n, m, p, and s.

- This form has the correct range.
- This form is K-equivariant.
- This form is consistent with the known form of the map $\phi^{(q)}_{n,m,p}$ in terms of the powers of a, b, c, d.

Thanks for listening!

Poisson and Fourier transforms for tensor products and an overalgebra,
F. Molchanov,
Geometric methods in physics (2015), 195–203

Inversion of Rankin-Cohen operators via Holographic Transform,
T. Kobayashi, M. Pevzner
Annales de l'Institut Fourier, Tome 70 (2020) no. 5, pp. 2131–2190

$p = m$ case

When $p = m$, we have:

$$\phi^{(q)}_{n,m,m}(a^l b^k) = \frac{[\ell]![m]!}{[n-m]!} \sum_{v=0}^{m} \sum_{i=v}^{\psi_2(k+v)} q^{-i(-m+i+1)+(i-n+k+v)(i-v)}(-1)^i \times \frac{[k]![n-i]!}{[v+k-i]![i-v]![m-i]![n-k-v]![v]!} a^{\ell+m-v} b^{n-\ell-m+v} c^v d^{m-v}$$

If we additionally assume $\ell \le n - 2m$:

$$\tilde{\phi}^{(q)}_{n,m,m}(a^l b^k) = \frac{[l]![m]!}{[n-m]!} \sum_{v=0}^{m} \sum_{i=v}^{m} q^{-i(-m+i+1)+(i-n+k+v)(i-v)}(-1)^i \times \frac{[k]![n-i]!}{[v+k-i]![i-v]![m-i]![n-k-v]![v]!} a^{l+m-v} b^{n-l-m+v} c^v d^{m-v}$$

Example of Calculations

When $p = m = 2$, the first ($v = 0$) coefficient of $\tilde{\phi}^{(q)}_{n,m,m}(a^l b^k)$ is given by:

$$\frac{[l]![2]!}{[n-2]!}\sum_{i=0}^{2}(-1)^i q^{-i(i-1)-2+k+(-l-2+i)(i)}\frac{[n-i]![k]!}{[2-i]![k-i]![i]![n-k]!}$$

$$= \frac{[l]![2]!}{[n-2]!}\left(q^{-2+k}\frac{[n]![k]!}{[2]![k]![n-k]!} - q^{k-l-3}\frac{[n-1]![k]!}{[k-1]![n-k]!} + q^{k-4-2l}\frac{[n-2]![k]!}{[k-2]![2]![n-k]!}\right)$$

$$= \frac{[n-2-k]!}{[n-2]!}\left(q^{-2+k}\frac{[n]!}{[n-k]!} - q^{2k-n-1}\frac{[2][n-1]![k]!}{[k-1]![n-k]!} + q^{3k-2n}\frac{[n-2]![k]!}{[k-2]![n-k]!}\right)$$

$$= q^{-2+k}\frac{[n][n-1]}{[n-k][n-k-1]} - q^{2k-n-1}\frac{[2][n-1][k]}{[n-k][n-k-1]} + q^{3k-2n}\frac{[k][k-1]}{[n-k][n-k-1]}$$

$$= \frac{1}{[n-k][n-k-1]}(q^{-2+k}[n][n-1] - q^{2k-n-1}[2][n-1][k] + q^{3k-2n}[k][k-1])$$

$$= \frac{1}{[n-k][n-k-1]}(q^{-2+k}\frac{(q^n-q^{-n})(q^{n-1}-q^{-n+1})}{(q-q^{-1})^2}$$

$$- q^{2k-n-1}\frac{(q^2-q^{-2})(q^{n-1}-q^{-n+1})(q^k-q^{-k})}{(q-q^{-1})^3} + q^{3k-2n}\frac{(q^k-q^{-k})(q^{k-1}-q^{-k+1})}{(q-q^{-1})^2})$$

Example of Calculations

$$
\begin{aligned}
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{(q-q^{-1})(q^{2n-3+k}-q^{k-3}-q^{k-1}+q^{-2n-1+k})}{(q-q^{-1})^3} \\
&- \frac{(q^{2k}-q^{2k-2n+2}-q^{2k-4}+q^{2k-2n-2})(q^{k}-q^{-k})}{(q-q^{-1})^3} \\
&+ \frac{(q^{1}-q^{-1})(q^{5k-2n-1}-q^{3k-2n-1}-q^{3k-2n+1}+q^{k-2n+1})}{(q-q^{-1})^3}\Big) \\
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{q^{2n-2+k}-q^{k-2}-q^{k}+q^{-2n+k}-q^{2n-4+k}+q^{k-4}+q^{k-2}-q^{-2n-2+k}}{(q-q^{-1})^3} \\
&- \frac{q^{3k}-q^{3k-2n+2}-q^{3k-4}+q^{3k-2n-2}-q^{k}+q^{k-2n+2}+q^{k-4}-q^{k-2n-2}}{(q-q^{-1})^3} \\
&+ \frac{q^{5k-2n}-q^{3k-2n}-q^{3k-2n+2}+q^{k-2n+2}-q^{5k-2n-2}+q^{3k-2n-2}+q^{3k-2n}-q^{k-2n}}{(q-q^{-1})^3}\Big)
\end{aligned}
$$

Example of Calculations

$$
\begin{aligned}
&= \frac{1}{[n-k][n-k-1]}\Big(\frac{q^{2n-2+k} - q^{k} + q^{-2n+k} - q^{2n-4+k} + q^{k-4} - q^{-2n-2+k}}{(q-q^{-1})^3} \\
&- \frac{q^{3k} - q^{3k-2n+2} - q^{3k-4} + q^{3k-2n-2} - q^{k} + q^{k-2n+2} + q^{k-4} - q^{k-2n-2}}{(q-q^{-1})^3} \\
&+ \frac{q^{5k-2n} - q^{3k-2n+2} + q^{k-2n+2} - q^{5k-2n-2} + q^{3k-2n-2} - q^{k-2n}}{(q-q^{-1})^3}\Big) \\
&= \frac{1}{[n-k][n-k-1]}\left(\frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q-q^{-1})^3}\right) \\
&= \frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q^{n-k} - q^{-n+k})(q^{n-k-1} - q^{-n+k+1})(q-q^{-1})} \\
&= \frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{(q^{2n-2k-1} - q^{-1} - q^{1} + q^{-2n+2k+1})(q-q^{-1})} \\
&= \frac{1}{q^{2-3k}}\left(\frac{q^{2n-2+k} - q^{2n-4+k} - q^{3k} + q^{3k-4} + q^{5k-2n} - q^{5k-2n-2}}{q^{2n-2+k} - q^{3k} + q^{5k-2n} - q^{2n+k-4} + q^{3k-4} - q^{-2n+5k-2}}\right) \\
&= q^{3k-2}
\end{aligned}
$$

$p = m$ case

By directly computing coefficients in the $p = m$ case we obtained a conjectural form for this specific example of a symmetry breaking operator.

Lemma

Fix $0 \leq p \leq m \leq n$ and let:

$$T, T' : V^{(q)}_{n+m-2p} \to V^{(q)}_n \otimes V^{(q)}_m$$

be two nonzero intertwiners between modules in the quantum Clebsch-Gordan formula. Then $\lambda T' = T$ must hold for some $\lambda \in \mathbb{C}$.

Proposition (ES)

The map:

$$\begin{aligned} \tilde{\phi}^{(q)}_{n,m,m} : \mathbb{C}_q[a,b]_{(n-m)} &\to \mathbb{C}_q[a,b,c,d]_{(n,m)} \\ f(a,b) &\mapsto (ac - q^{m-n-1}bd)^m \sigma^m_b(f(a,b)) \end{aligned}$$

intertwines $\mathbb{C}_q[a,b]_{(n-m)}$ and $\mathbb{C}_q[a,b,c,d]_{(n,m)}$ as $U_q(\mathfrak{sl}(2,\mathbb{C}))$-modules. Thus, $\tilde{\phi}^{(q)}_{n,m,m}$ is a quantum Poisson transform in the $p = m$ case.