FUSION SYSTEMS OF BLOCKS OF FINITE GROUPS OVER ARBITRARY FIELDS

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Overview

- Reviewing block theory and related notions
- Fusion systems, saturated fusion systems, block fusion systems
- Our observations

Throughout:

- G is a finite group,
- p is a prime dividing the order of G,
- k is a field of characteristic p,
- kG is the group algebra (k-vector space with G as its basis)
- $Z(kG)$ is the center of kG

Definition

- An **idempotent** of any ring R, is an non-zero element $e \in R$ such that $e^2 = e$.
- Two idempotents $e, f \in R$ are called **orthogonal** if $ef = 0 = fe$.
- An idempotent $e \in R$ is called **primitive** if it cannot be written as $e = e_1 + e_2$ with e_1, e_2 orthogonal idempotents of R.
- A **primitive decomposition** of 1_R is a set $I = \{e_1, e_2, \cdots, e_n\}$ of pairwise orthogonal and primitive idempotents with $e_1 + e_2 + \cdots + e_n = 1_R$.

Definition

- A **block idempotent** b of kG is a primitive idempotent of $Z(kG)$.
- The algebra $B := kGb$ is called a **block** of kG and it is an indecomposable as k-algebra and similarly (kG, kG)-bimodule.
- Let $\{b_1, b_2, \cdots, b_n\}$ be a primitive decomposition of 1 in $Z(kG)$. Denote $kGb_i := B_i$. Then, $kG = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is called the **block decomposition** of kG.

(G-algebra is a k-algebra with a G-action.)

Relative Trace Map

- Let A be a G-algebra over k, a field of characteristic p. For $H \leq G$, let $A^H := \{a \in A \mid {}^h a = a \text{ for all } h \in H\}$.
- Note that if $L \leq H \leq G$, we have $A^H \subseteq A^L$.
- The **relative trace map** $\mathrm{Tr}_L^H : A^L \to A^H$ is defined by $a \mapsto \sum_{h \in [H/L]} {}^h a$.
- $A_L^H := \mathrm{Im}(\mathrm{Tr}_L^H)$.

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Brauer homomorphism

- Let A be a G-algebra over k, a field of characteristic p. For $H \leq G$, let $A^H_{<H}$ be the sum of all relative traces A^H_L with $L < H$.
- The **Brauer quotient** is $A(H) := A^H / A^H_{<H}$.
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Remark

If A is the group algebra kG, then the Brauer map is just the k-linear projection $\mathrm{Br}_P^{kG} : (kG)^P \twoheadrightarrow kC_G(P)$,

$$\sum_{g \in G} a_g g \mapsto \sum_{g \in C_G(P)} a_g g.$$

Definition (Brauer Pair)

- **A kG-Brauer pair** is a pair (P, e) where P is a p-subgroup of G and e is a block idempotent of $kC_G(P)$.
- If i is an idempotent of $(kG)^P$, we say i is **associated** to (P, e) if $e\mathrm{Br}_P^{kG}(i) = \mathrm{Br}_P^{kG}(i)e = \mathrm{Br}_P^{kG}(i) \neq 0$.

Definition

Let (Q, f) and (P, e) be kG-Brauer pairs. We say that (Q, f) is **contained** in (P, e) and write $(Q, f) \leq (P, e)$ if $Q \leq P$ and if any primitive idempotent i of $(kG)^P$ which is associated to (P, e) is also associated to (Q, f).

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Theorem

Let (P, e) be a kG-Brauer pair and let $Q \leq P$.

(a) There exists a unique block idempotent f of $kC_G(Q)$ such that $(Q, f) \leq (P, e)$.

(b) Inclusion of kG-Brauer pairs is a transitive relation.

Remark

The set of kG-Brauer pairs is a G-poset via the map sending an kG-Brauer pair (P, e) and $x \in G$ to the kG-Brauer pair ${}^x(P, e) = ({}^xP, {}^xe)$.

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Definition (b-Brauer pair)

Let b be a block idempotent of kG. A **b-Brauer pair** is an kG-Brauer pair (P, e) such that $\mathrm{Br}_P^{kG}(b)e \neq 0$.

Remark

Let $(Q, f) \leq (P, e)$ be kG-Brauer pairs. If (P, e) is a b-Brauer pair, then so are (Q, f) and ${}^x(P, e)$ for any $x \in G$.

Notation

We denote by $\mathcal{BP}(kG)$ the set of kG-Brauer pairs and by $\mathcal{BP}(kG, b)$ the set of b-Brauer pairs.

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Definition

Let b a block idempotent of kG. A subgroup P of G, minimal with the property that $b \in \mathrm{Tr}_{\mathrm{P}}^{\mathrm{G}}((\mathrm{kG})^{\mathrm{P}})$ is called a **defect group** of the block idempotent b and of the block algebra kGb.

- The defect groups of kGb form a single G-conjugacy class of p-subgroups of G.

Theorem

(a) The maximal elements in $\mathcal{BP}(kG, b)$ with respect to $\leq$ form a single G-orbit.

(b) For $(P, e) \in \mathcal{BP}(kG, b)$ the following are equivalent:

(i) (P, e) is a maximal element in $\mathcal{BP}(kG, b)$.

(ii) P is a defect group of kGb.

(iii) P is a maximal among all p-subgroups of G with the property $\mathrm{Br}_P(b) \neq 0$.

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Fusion systems

Notation

For subgroups Q and R of G,

- $\mathrm{Hom}_G(Q, R)$ denotes the set of all group homomorphisms $\phi : Q \to R$ with the property that there exists $g \in G$ with $\phi(x) = c_g(x)$ for all $x \in Q$.
- We set $\mathrm{Aut}_G(Q) := \mathrm{Hom}_G(Q, Q)$.

Definition

Fix a finite group G. Let $P \in \mathrm{Syl}_p(G)$. The **fusion category** of G over P is the category $\mathcal{F}_P(G)$ whose objects are the subgroups of P, and the morphism sets are, for all subgroups Q and R of P, $\mathrm{Hom}_G(Q, R)$.

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The (abstract) fusion system

Definition

Let P be a finite p-group. A **fusion system over** P is a category $\mathcal{F}$ whose objects are the subgroups of P, and for any $Q, R \leq P$, the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ has the following properties:

- $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R) \subseteq \mathrm{Inj}(Q, R)$
- For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, the group isomorphism $Q \to \varphi(Q)$, $u \mapsto \varphi(u)$, and its inverse are morphisms in $\mathcal{F}$.

Example

•$\mathcal{F}_P(G)$ where $P \in \mathrm{Syl}_p(G)$.
•$\mathcal{F}_{(P,e_P)}(kGb)$, fusion system of a block b of kG over a p-group P, where k is an arbitrary field of prime characteristic p.

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Definition

Let $\mathcal{F}$ be a fusion system over a finite p-group P.

- A subgroup Q of P is called **fully $\mathcal{F}$-centralized** if $|C_P(Q)| \geq |C_P(R)|$ for any subgroup R of P which is $\mathcal{F}$-isomorphic to Q.
- A subgroup Q of P is called **fully $\mathcal{F}$-normalized** if $|N_P(Q)| \geq |N_P(R)|$ for any subgroup R of P which is $\mathcal{F}$-isomorphic to Q.

Definition

Let $\mathcal{F}$ be a fusion system over a p-group P and $\varphi : Q \to R$ be an isomorphism in $\mathcal{F}$. We define

$$N_\varphi := \{y \in N_P(Q) \mid \exists z \in N_P(R) \text{ s.t. } \varphi \circ c_y = c_z \circ \varphi : Q \to R\}.$$

Note that $QC_P(Q) \leq N_\varphi \leq N_P(Q)$.

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Note that $QC_P(Q) \leq N_\varphi \leq N_P(Q)$.

Definition

A fusion system $\mathcal{F}$ over a p-group P is called **saturated** if the following two conditions hold:

(i) **Sylow axiom:** $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(ii) **Extension axiom:** For every $Q \leq P$, and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized, there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, P)$ whose restriction to Q equals to φ.

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$\mathcal{F}_P(G)$ is saturated where $P \in Syl_p(G)$.

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Example

$\mathcal{F}_P(G)$ is saturated where $P \in Syl_p(G)$.

Why are the saturated fusion systems nice?

Alperin's Fusion Theorem

Let $\mathcal{F}$ be a saturated fusion system over a p-group P. Then,

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) \mid Q = P \text{ or Q is } \mathcal{F}\text{-centric} \rangle_P.$$

Definition

A subgroup $Q \leq P$ is $\mathcal{F}$-**centric** if $C_P(R) = Z(R)$ for all R which are $\mathcal{F}$-isomorphic to Q.

Notation: Let b be a block of kG and (P, e_P) be a maximal b-Brauer pair. For each $Q \leq P$, let e_Q denote the unique block of $kC_G(Q)$ such that $(Q, e_Q) \leq (P, e_P)$.

Definition

The fusion system of a block kGb over (P, e_P) is the category $\mathcal{F}_{(P,e_P)}(kGb)$ whose objects are the subgroups of P and which has morphism sets, for subgroups Q and R of P,

$$\{\varphi \in \mathrm{Hom}(Q, R) : \varphi = c_g \text{ for some } g \in G \text{ s.t. } {}^g(Q, e_Q) \leq (R, e_R)\}.$$

Theorem

Let (P, e_P) be a maximal b-Brauer pair and suppose that k is a splitting field for $kC_G(P)e_P$, i.e. for every simple $kC_G(P)e_P$-module V one has a k-algebra isomorphism $\mathrm{End}_{kC_G(P)e_P}(V) \cong k$. Then, the category $\mathcal{F}_{(P,e_P)}(kGb)$ is saturated.

Remark

If k is not a splitting field for $kC_G(P)e_P$, there are examples in which the corresponding block fusion system **fails** to be saturated.

Example

Let $p = 2$, $k = \mathbb{F}_2$ and $G = D_{24} = (C_3 \times C_4) \rtimes C_2$. Let g be the generator of C_3.

- $b := g + g^2$ is a block idempotent of $\mathbb{F}_2 G$,
- $(P, e) := (C_4, b)$ is a maximal $(\mathbb{F}_2 G, b)$-Brauer pair,
- One has $\mathrm{Aut}_P(P) = \{1\}$,
- $\mathrm{Aut}_{\mathcal{F}_{(P,e)(\mathbb{F}_2 Gb)}}(P) \cong C_2$.
- Then $\mathrm{Aut}_P(P) \notin Syl_p(\mathrm{Aut}_{\mathcal{F}_{(P,e)(\mathbb{F}_2 Gb)}}(P))$. Hence Sylow axiom fails and the block fusion system $\mathcal{F}_{(P,e)(\mathbb{F}_2 Gb)}$ is **not** saturated.

Throughout: Let L/K be a Galois extension of finite fields of characteristic p, prime and $\Gamma := Gal(L/K)$.

- Γ acts via K-algebra automorphism on LG and also on $Z(LG)$ by applying $\gamma \in \Gamma$ to the coefficients of an element in LG.
- Γ permutes the block idempotents of LG.
- Brauer homomorphism commutes with Γ-action.

Proposition

(i) If b is a block of LG, then ${}^{\Gamma}b := \mathrm{tr}(b) = \sum_{\gamma \in \Gamma/\mathrm{stab}_{\Gamma}(b)} \gamma(b)$ is a block of KG.

(ii) There is a bijective correspondence between $\mathrm{Bl}(LG)/\Gamma \longleftrightarrow \mathrm{Bl}(KG)$ induced by $b \mapsto {}^{\Gamma}b$.

(iii) If ${}^{\Gamma}b$ and b are corresponding blocks of KG and LG, then they have the same defect groups.

Notation: $\leq_K$ and $\leq_L$ denote the poset structures of $\mathcal{BP}(KG)$ and $\mathcal{BP}(LG)$, respectively.

Proposition

For $(Q, f), (P, e) \in \mathcal{BP}(LG)$ with $Q \leq P$, the following are equivalent:

(i) $(Q, f) \leq_L (P, e)$ in $\mathcal{BP}(LG)$.

(ii) $(Q, {}^{\Gamma}f) \leq_K (P, {}^{\Gamma}e)$ in $\mathcal{BP}(KG)$.

Lemma

Let $\mathcal{BP}(LG, b)$ denote the set of (LG, b)-Brauer pairs and similarly $\mathcal{BP}(KG, {}^{\Gamma}b)$ for $(KG, {}^{\Gamma}b)$-Brauer pairs. Then, we have surjective G-poset map

$$\mathcal{BP}(LG, b) \twoheadrightarrow \mathcal{BP}(KG, {}^{\Gamma}b) \text{ given by } (Q, f) \mapsto (Q, {}^{\Gamma}f).$$

Lemma

Let (P, e) be maximal in $\mathcal{BP}(LG, b)$ then $(P, {}^{\Gamma}e)$ be maximal in $\mathcal{BP}(KG, {}^{\Gamma}b)$. There exists an embedding

$$\mathcal{I} : \mathcal{F}_{(P,e)}(LGb) \hookrightarrow \mathcal{F}_{(P,{}^{\Gamma}e)}(KG^{\Gamma}b).$$

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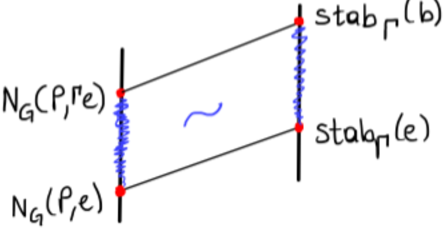
Idea

(P, e) is a maximal (LG, b)-Brauer pair.
$(P, {}^{\Gamma}e)$ is a maximal $(KG, {}^{\Gamma}b)$-Brauer pair.



Observation: $N_G(P, {}^{\Gamma}e)/N_G(P, e) \cong \mathrm{stab}_{\Gamma}(b)/\mathrm{stab}_{\Gamma}(e)$ is a cyclic group.

Main Theorem

Theorem

Let b be a block of LG, and (P, e) a maximal (LG, b)-Brauer pair. Let $g_0 \in N_G(P, {}^\Gamma e)$ be such that $\langle g_0 N_G(P, e) \rangle = N_G(P, {}^\Gamma e)/N_G(P, e)$ and set $\sigma := c_{g_0} \in \mathrm{Aut}(P)$. Then,

$$\mathcal{F}_{(P, {}^\Gamma e)}(KG^\Gamma b) = \langle \mathcal{F}_{(P,e)}(LGb), \sigma \rangle.$$

Consequences of the Main Theorem

Proposition

Using the same notation as before.

(i) $Q \leq P$ is fully $\mathcal{F}_{(P,e)}(LGb)$-centralized(normalized) if and only if Q is fully $\mathcal{F}_{(P,{}^{\Gamma}e)}(KG^{\Gamma}b)$-centralized(normalized).

(ii) $Q \leq P$ is $\mathcal{F}_{(P,e)}(LGb)$-centric if and only if Q is $\mathcal{F}_{(P,{}^{\Gamma}e)}(KG^{\Gamma}b)$-centric.

Theorem

Using the same notation as before, $\mathcal{F}_{(P,{}^{\Gamma}e)}(KG^{\Gamma}b)$ *is saturated if and only if* $\mathcal{F}_{(P,e)}(LGb)$ *is saturated and* $[\mathrm{stab}_{\Gamma}(b) : \mathrm{stab}_{\Gamma}(e)]$ *is not divisible by* p*.*

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Thank You