Von Neumann Algebras, Subfactors and Knots

III. The Jones Tower & Applications

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GAG seminar

April, 16 2021

1. Knots and Their Invariants
2. The Jones Polynomial
3. Summary of Last Week
4. The Basic Construction and the Jones Tower
5. Temperley-Lieb Algebras, Braids and Knots

This is Week 12 of the semester...

Today, we will be reading from:

The Jones polynomial for dummies.

Vaughan Jones[2]

February 12, 2014

[2] supported by NSF under Grant No. DMS-XYZ

The text and illustrations in the first two sections are reproduced verbatim from this document, available at:

`https://math.berkeley.edu/~vfr/jonesakl.pdf`

A knot is a smooth closed curve in three dimensional space $\mathbb{R}^3$. As such, it is an object of topology, two knots are "the same" if one can be obtained by the other by smooth deformations of $\mathbb{R}^3$.

A link is a disjoint union of several knots.

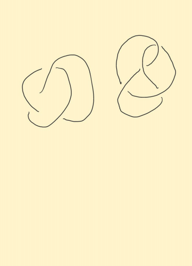
Two fortunate features make the theory of knots particularly accessible and allow us to actually forget the analysis underlying the word "smooth".

The first is that *knots are perfectly adequately modeled by bits of string and the "smooth deformations" are modeled by manual manipulation of the strings (without ever cutting them).*

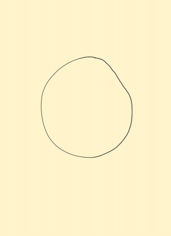
Thus the question:

Is this knot the same as that knot?

can be approached quite experimentally by tying the knots and seeing if the first can be manipulated to look like the other.



One immediately meets the necessity of mathematical proof: how can one be sure that a little more manipulation would not have turned the first knot (the trefoil) into the second (the figure 8):



...or indeed that either of them can be converted into the unknot?

Whatever sense one may attach to the word "topology", knot theory fits into topology and one should search for ways of distinguishing knots from topology.

Moreover the second fortunate feature of knot theory is now visible: although knots are inherently 3 dimensional, they can be faithfully represented by two dimensional pictures such as the ones we have seen above, all the 3 dimensionality being reduced to whether the crossings in the picture are over or under.

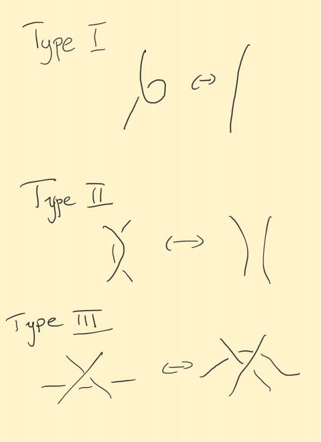
And the fundamental question of knot theory becomes:

When do two pictures of knots represent the same knot?

When do two pictures of knots represent the same knot?

This question was answered in the early twentieth century by Reidemeister. There are three "*Reidemeister moves*" which act locally on pictures, only changing that part included in the move.

We draw the Reidemeister moves below:



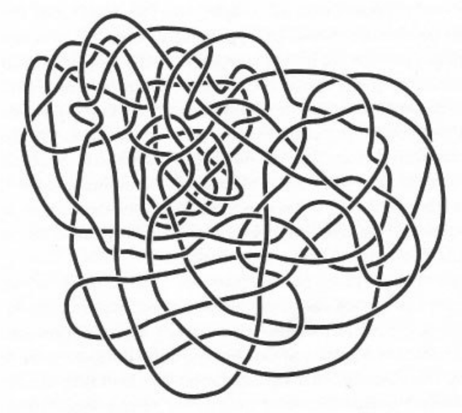
The theorem is that any two pictures of the same knot can be obtained from one another by two dimensional deformations (called isotopies) and Reidemeister moves.

It is in fact not too hard to convince oneself of this but the proof is really a four dimensional thing-one considers a "movie" of pictures and one must arrange for the simplest possible things to happen when crossings meet one another.

The Reidemester moves reduce knots to objects of planar combinatorics!

This does not however necessarily simplify matters. But one can search for *combinatorial formulae that don't change under the Reidemeister moves.*

'This does not however necessarily simplify matters.'



What is the Jones Polynomial?

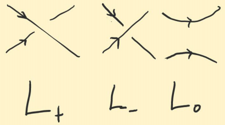
The Jones polynomial is an assignment of Laurent polynomials $V_L(t)$ in the variable $\sqrt{t}$ to oriented links L subject to the following three axioms:

(a) Two equal links have the same polynomial.

(b) The polynomial of the unknot is equal to 1.

(c) The *skein relation*:

if three links L_+, L_- and L_0 have pictures which are identical apart from within a region where they are as this:



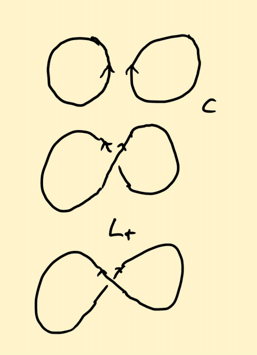
then:

$$\boxed{t^{-1} V_{L_+} - t V_{L_-} = \left(t^{1/2} - t^{-1/2}\right) V_{L_0}}.$$

This is less a definition than a calculational method.

There is no guarantee at this stage that such an invariant exists. I want to convince you though, right away, that the "skein" formula suffices to calculate $V_L(t)$, inductively on all links.

We begin with two unlinked circles. Call that link C. Then consider the following picture:



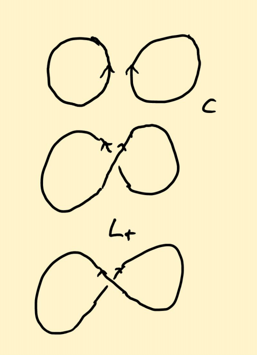
Clearly we may apply axiom (c) with both L_+ and L_- being the unknot and L_0 being C.

So we have:

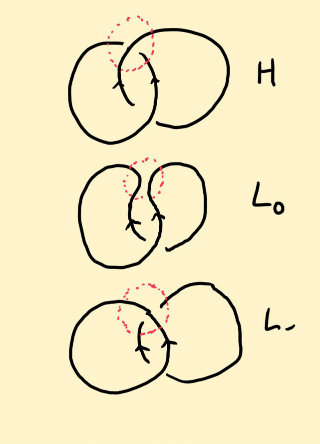
$$t^{-1}V_{L_+} - tV_{L_-} = \left(t^{1/2} - t^{-1/2}\right)V_C$$

which gives

$$V_C = \frac{t^{-1} - t}{t^{1/2} - t^{-1/2}} = -\left(t^{1/2} + t^{-1/2}\right).$$



Now consider the link called H in the following picture (sometimes called the Hopf link):



Again by axiom (c) we have:

$$t^{-1} V_H - t V_{L_-} = \left(t^{1/2} - t^{-1/2}\right) V_{L_0}$$

which by our previous calculation gives

$$t^{-1} V_H = t V_C + \left(t^{1/2} - t^{-1/2}\right)$$

hence

$$V_H = -t^{1/2}\left(1 + t^2\right).$$

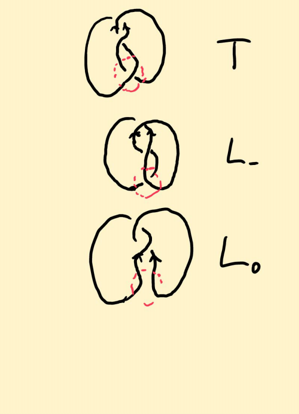
Now finally we can calculate the polynomial of the trefoil:

We have:

$$t^{-1}V_T - tV_{L_-} = \left(t^{1/2} - t^{-1/2}\right)V_{L_0}$$

and since L_- is the unknot and L_0 is now H:

$$\begin{aligned} t^{-1}V_T &= t + \left(t^{1/2} - t^{-1/2}\right) \times \left(-t^{1/2}\left(1 + t^2\right)\right) \\ &= t + t^3 - t^4. \end{aligned}$$



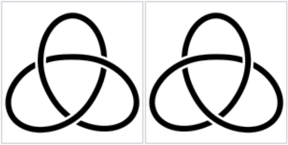
It should now be fairly obvious that the axioms for the Jones polynomial suffice to calculate it. The point is that any knot/link can certainly be untied by changing enough crossings.

Some observations are immediate:

- If L and L' are mirror images of one another, then

$$V_{L'}(t) = V_L(t^{-1})$$

- $\rightsquigarrow$ The mirror image of the trefoil is not the same as the trefoil:



- The polynomial of a link with an odd number of components is a (Laurent) polynomial in t and the polynomial of a link with an even number of components is $t^{1/2}$ times a (Laurent) polynomial in t
- The polynomial of the "connected sum" of two knots is the product of their polynomials:

$$V_{L\#L'} = V_L \cdot V_{L'}.$$

And now for something completely different...

Von Neumann Algebras and Factors

Definition

- A *von Neumann algebra* is a $*$-subalgebra M of $\mathfrak{B}(\mathcal{H})$ that coincides with its bicommutant:
$$M = M''$$
- A von Neumann algebra $M \subset \mathfrak{B}(\mathcal{H})$ is called a *factor* if
$$M \cap M' = \mathbb{C}.1_{\mathfrak{B}(\mathcal{H})}$$

- Examples of factors include: $M_n(\mathbb{C})$, $\mathfrak{B}(\mathcal{H})$, $L(\Gamma)$ for Γ icc group...
- Factors can be of type:
  - I_n or I_∞ (classified by dimension)
  - II_1 or II_∞ (with $II_\infty \simeq II_1 \otimes \mathfrak{B}(\mathcal{H})$)
  - III_λ (with $\lambda \in [0, 1]$)

We focus on II_1 factors.

Projections and Traces

Definition

A *projection* in a von Neumann algebra M is an element $p \in M$ such that:

$$p^* = p = p^2.$$

The type of a factor depends on the existence of 'small' projections.

Definition

A linear functional τ on a von Neumann algebra M is called:

- *positive* if $\tau(x^*x) \geq 0$ for all $x \in M$
- *faithful* if $\tau(x^*x) = 0 \quad \Rightarrow \quad x = 0$
- *a state* if $\tau(1) = 1$
- *tracial* if $\tau(xy) = \tau(yx)$ for all $x, y \in M$

A tracial state is called a *trace*.

II_1 factors are characterized among ∞-dim'l factors by the existence of a norm-continous, faithful trace, generally denoted by τ.

Standard Form of a II_1 factor

Let M be a factor with continuous and faithful trace τ. The pairing

$$\langle x, y\rangle = \tau(y^* x) \in \mathbb{C}$$

defines an inner product on M.
Let $L^2(M,\tau)$ be the associated completion of M. There is an embedding:

$$\begin{array}{ccc} M & \longrightarrow & L^2(M) \\ x & \longmapsto & \hat{x} \\ 1 & \longmapsto & \Omega \end{array}.$$

M is represented on the Hilbert space $L^2(M)$ by considering:

$$\pi_\tau(x)\hat{y} = \widehat{xy} := x\hat{y}$$

and extending it to a morphism

$$\pi_\tau : M \longrightarrow \mathfrak{B}(L^2(M)).$$

This representation is called the *standard form* of M and we have:

$$\hat{x} = \widehat{x1} = x\hat{1} = x\Omega$$

so that $\tau(x) = \langle x\Omega, \Omega\rangle$.

Representations and von Neumann Dimension

A *representation* of a factor M is a Hilbert space $\mathcal{H}$ with a structure of M-module (M acts on $\mathcal{H}$ by bounded operators).

Theorem

Any representation $\mathcal{H}$ of a II_1 factor M is equivalent to $p\big(L^2(M) \otimes \ell^2(\mathbb{N})\big)$ for some projection $p = vv^*$ in $(M \otimes 1)'$.

Idea: the size of the representation $\mathcal{H}$ is measured by the trace of p.

Definition

The *M-dimension* of the M-module $\mathcal{H}$ is

$$\dim_M \mathcal{H} \doteqdot \mathrm{tr}(vv^*) \in [0, \infty].$$

Properties of the von Neumann Dimension

Let M be a II_1 factor.

- $\dim_M \mathcal{H} \in [0, \infty]$ and all values occur.
- Direct sums:

$$\dim_M \bigoplus_{j \in J} \mathcal{H}_j = \sum_{j \in J} \dim_M \mathcal{H}_j.$$

- If $p \in M'$ is a projection then:

$$\dim_M L^2(M)p = \tau(p).$$

- Representations of M are classified by their M-dimension:

$$\dim_M \mathcal{H} = \dim_M \mathcal{K} \quad \Leftrightarrow \quad \mathcal{H} \simeq \mathcal{K}.$$

The Jones Index

Let $N \subset M$ be II_1 factors. Then N is represented on $L^2(M)$.

Definition

The *Jones Index* of the subfactor $N \subset M$ is

$$[M : N] \doteqdot \dim_N L^2(M).$$

Example: if $M = M_k(\mathbb{C}) \otimes N \simeq M_k(N)$, then $[M : N] = k^2$.

Proposition

If $N \subset M$ is represented on $\mathcal{H}$ with $\dim_N \mathcal{H} < \infty$, then

$$[M : N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.$$

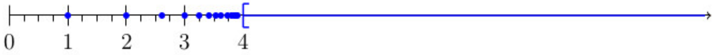
The Jones Index Theorem

$\rightsquigarrow$ Beside $1, 4, 9, \ldots$ what possible values can $[M : N]$ take?

Theorem (Jones, 1983)

Let $N \subset M$ be a II_1 subfactor. The possible values of $[M : N]$ are given by:

$$\left\{4\cos^2\left(\frac{\pi}{n+2}\right) \ : \ n \geq 1\right\} \cup [4, \infty).$$



Questions:

- How to prove that some numbers cannot be indices?
- How to construct subfactors $N \subset M$ with prescribed index $[M : N]$?
- What does it have to do with knot theory?

The Jones Projection

Let $N \subset M$ be II_1 factors. Then,

- $M' \subset N'$,
- $L^2(N)$ is a subspace of $L^2(M)$.

Proposition

Let $e_N \in \mathfrak{B}(L^2(M))$ be the orthogonal projection onto $L^2(N)$.
Then, for $x \in M$,

$$xe_N = e_N x \quad \Leftrightarrow \quad x \in N$$

and

$$N' = (M' \cup \{e_N\})'' .$$

Remark: e_N gives the *'difference'* between M' and N'.

The Basic Construction

Let $N \subset M$ be II_1 factors and

$$e_N : L^2(M) \longrightarrow L^2(N) \subset L^2(M)$$

the Jones projection.

Let M_1 be the von Neumann algebra $\langle M, e_N \rangle$ generated by M and e_N in $\mathfrak{B}(L^2(M))$:

$$M_1 := \{M \cup \{e_N\}\}'' \subset \mathfrak{B}(L^2(M)).$$

Proposition

The v.N. algebra M_1 is a factor if and only if $[M : N] < \infty$, in which case

$$[M_1 : M] = [M : N].$$

In addition, $\tau_{M_1}(e_N) = [M : N]^{-1}$.

Application: combining this with

$$[P : R] = [P : Q] \cdot [Q : R]$$

implies that there can be no subfactor $N \subset M$ with $1 < [M : N] < 2$.

The Jones Tower

Let $N \subset M$ be II_1 factors with $[M : N] < \infty$. The basic construction can be iterated, by letting

$$M_{-1} := N \qquad , \qquad M_0 : M$$

and, for $i \geq 1$,

$$M_{i+1} := \{M_i\,,\, e_{M_{i-1}}\}\,.$$

Denoting $e_i := e_{M_{i-2}}$, one thus obtains:

The Jones Tower of Factors

$$N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} \ldots \overset{e_{i-1}}{\subset} M_{i-1} \overset{e_i}{\subset} M_i \overset{e_{i+1}}{\subset} \ldots$$

Proposition

Let $\lambda = [M : N]^{-1}$. The Jones projections satisfy:

- $e_i^2 = e_i = e_i^*$
- $e_i e_j = e_j e_i$ if $|i - j| \geq 2$
- $e_i e_{i\pm 1} e_i = \lambda e_i$

And now for something completely different(?)

Temperley-Lieb Algebras

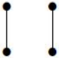
Definition

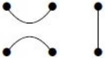
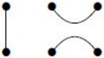
For $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, the *Temperley-Lieb* algebra $TL_{n+1}(\lambda)$ is the unital algebra with generators $e_1, \ldots, e_n$ and relations

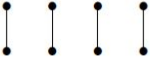
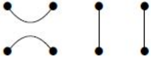
- $e_i^2 = e_i$
- $e_i e_j = e_j e_i$ if $|i - j| \geq 2$
- $e_i e_{i\pm 1} e_i = \lambda e_i$

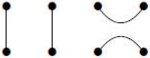
These algebras have a diagrammatic representation.

TL_1: $1 =$ 

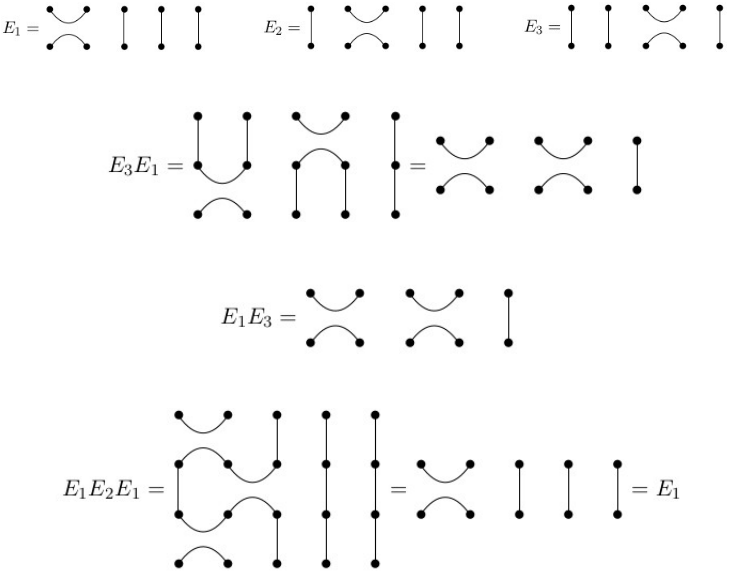
TL_2: $1 =$  , $E_1 =$ 

TL_3: $1 =$  , $E_1 =$  , $E_2 =$ 

TL_4: $1 =$  , $E_1 =$  ,

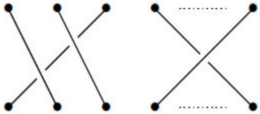
$E_2 =$  , $E_3 =$ 

Composition in TL_n

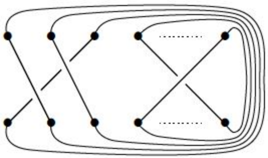


From Temperley-Lieb Algebras to Knots (sketch)

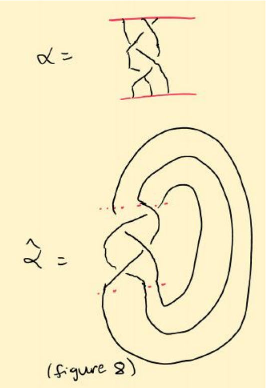
Braid groups are closely related to TL_n algebras.

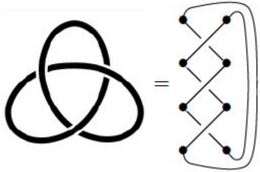


Braids can be closed to form links:



In fact, every link can be obtained by closing a braid (Alexander ’23):





The construction of the Jones Polynomial involves the representation theory of Temperley-Lieb algebras, Bratteli diagrams...but that is a story for another ~~day~~ semester.

Thank you.