Reality and Strong Reality in Finite Symplectic Groups

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GAG Seminar
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- Quick example: $(12)(123)(12) = (132) \in S_3$.

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  $sgs^{-1} = tgt^{-1} \implies gs^{-1}t = s^{-1}tg \implies t \in sC_G(g).$

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- Modding out by a subgroup:

  Let N be a normal subgroup of G. Then $gN \sim hN$ in $G/N \iff g$ is conjugate to an element of hN in G.

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  j is real with $ij(-i) = -j$, but j is not strongly real

Representation Theory

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- And given an F-representation ρ of G, the F-character of G afforded by ρ is the map $\chi : G \to F$ defined by $\chi(g) = \mathrm{tr}(\rho(g))$. The character χ is itself called irreducible if it is afforded by an irreducible representation.

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- In fact, $g \in G$ is real if and only if $\chi(g)$ is real for every complex-valued irreducible character χ of G,
  and the number of real-valued irreducible complex characters of G is equal to the number of real classes.

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Conjecture (Vinroot)

If G is a finite simple group of Lie type, then all real classes of G are strongly real if and only if all real-valued irreducible characters of G are afforded by a real representation.

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Theorem (Wonenburger, 1966)

All real classes of $GL(n, q)$ are strongly real.

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- Any choice of the form B results in the same group, up to isomorphism!

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- $Z(\mathrm{Sp}(V)) = \{\pm 1\}$

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- Fact: the groups $\mathrm{PSp}(2n, q)$ are simple except for $\mathrm{PSp}(2, 2)$, $\mathrm{PSp}(2, 3)$, $\mathrm{PSp}(4, 2)$

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- So it'll take some work to determine the real classes in $\mathrm{Sp}(2n,q)$

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- But we do get strong reality in $\mathrm{PSp}(2n, q)$ (for $q \equiv 1 \pmod 4$)!

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When $q \equiv 3 \pmod 4$, -1 is not a square in F, so we get situations like the following:

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(real but not strongly real in $\mathrm{Sp}(4, q)$ when $q \equiv 3 \pmod 4$)

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- Strong reality in the projective symplectic group?

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Conjecture

All real classes of $PSp(2n,q)$ are strongly real.

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For the rest of the way, assume that $q \equiv 3 \pmod{4}$.

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Proposition

The element $g \in Sp(2n, q)$ is real in $Sp(2n, q)$ if and only if it has no elementary divisors of form $(t \pm 1)^{2\ell}$, $\ell \geq 1$ appearing with odd multiplicity.

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$$\text{with } \Omega = \begin{pmatrix} & 1 & & & & & & \\ -1 & & & & & & & \\ & & & 1 & & & & \\ & & -1 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & & 1 \\ & & & & & & -1 & \end{pmatrix}$$

Progress

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- Elements with all eigenvalues ± 1

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Proposition

A real unipotent or negative unipotent element in $Sp(2n,q)$ is conjugate to its inverse by an element in $Sp(2n,q)$ whose square is -1.

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Proposition

A real unipotent or negative unipotent element in $Sp(2n, q)$ is conjugate to its inverse by an element in $Sp(2n, q)$ whose square is -1.

Proposition

A negative real element of $Sp(2n, q)$ whose elementary divisors are all of the form $(t \pm 1)^{\ell}$, $\ell \geq 1$ is conjugate to its negative inverse by an element in $Sp(2n, q)$ whose square is $+1$.

Progress

Theorem

All real unipotent classes of $PSp(2n, q)$ are strongly real in $PSp(2n, q)$.

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We would like this result for all real classes, and we are getting close!

Thank you!