Von Neumann Algebras, Subfactor and Knots

II. Subfactors

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GAG seminar

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1 Summary of Last Week

2 Factors of Type $II_1$

3 Representation Theory of $II_1$ Factors and the Jones Index
If $\mathcal{H}$ is a Hilbert space.

Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators from $\mathcal{H}$ to itself.

**Theorem**

If $T \in \mathcal{B}(\mathcal{H})$, there exists a unique operator $T^*$ such that:

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$$

for all $\xi, \eta$ in $\mathcal{H}$.

We are interested in subalgebras of $\mathcal{B}(\mathcal{H})$ that are stable under $T \mapsto T^*$. 
**C*-algebras**

**Definition**

A C*-algebra is a closed subalgebra $A \subset \mathcal{B}(H)$ that is stable under adjoint.

**Examples:**

- $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$
- $\mathcal{B}(H)$: bounded operators on $H$
- $\mathcal{K}(H)$: compact operators on $H$
- $A$ commutative $\Rightarrow A \simeq C_0(X)$.

I. Gelfand (1913 - 2009)
Commutants

The *commutant* of a subset $A \subset \mathcal{B}(\mathcal{H})$ is

$$A' := \{ y \in \mathcal{B}(\mathcal{H}) , \ xy = yx \text{ for all } x \in A \}.$$

The *bicommutant* of $A$ is

$$A'' = (A')'.$$

**Remark:** $A \subset A''$ for any $A$.

**Example:** let $A = M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$. Then

$$A' = \mathbb{C}.I_n \quad \text{ and } \quad A'' = M_n(\mathbb{C}) = A$$
Definition

A von Neumann algebra is a $\ast$-subalgebra $M$ of $\mathcal{B}(\mathcal{H})$ such that $M = M''$.

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$
- $\mathcal{B}(\mathcal{H})$: bounded operators on $\mathcal{H}$
- $M$ commutative $\Rightarrow M \simeq L^\infty(X, \mu)$. 

John von Neumann
(1903 - 1957)
A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is called a factor if

$$M \cap M' = \mathbb{C} \cdot 1_{\mathcal{B}(\mathcal{H})}$$

Any separable von Neumann algebra $M$ is isomorphic to a direct integral of factors:

$$M \simeq \int_{\mathcal{T}} M_t \, d\mu(t)$$
Let $\Gamma$ be a discrete group with group algebra

$$\mathbb{C}\Gamma = \left\{ \sum_{\gamma \in \Gamma} \alpha_{\gamma} \cdot \gamma \mid \alpha_{\gamma} \in \mathbb{C} \text{ and } \alpha_{\gamma} \neq 0 \text{ only for finitely many } \gamma \right\}.$$ 

Then $\mathbb{C}\Gamma$ acts on the Hilbert space $\ell^2(\Gamma)$ by left multiplication:

$$\mathbb{C}\Gamma \subset \mathcal{B}(\ell^2(\Gamma))$$

and the von Neumann algebra of $\Gamma$ is:

$$L(\Gamma) := \overline{\mathbb{C}\Gamma}^{SOT} = \overline{\mathbb{C}\Gamma}^{WOT} = \mathbb{C}\Gamma''$$

**Theorem**

$L(\Gamma)$ is a factor $\iff$ $\Gamma$ has infinite conjugacy classes.

**The Free Group Isomorphism Problem**

If $m \neq n$, is it the case that $L(F_m) \neq L(F_n)$? 

*Still open...*
A projection in a von Neumann algebra $M$ is an element $p \in M$ such that:

$$p^* = p = p^2.$$ 

We write $p \leq q$ if $p\mathcal{H} \subset q\mathcal{H}$.

- Two projection $p, q \in M$ are said equivalent ($p \sim q$) if there is a partial isometry $u \in M$ such that
  
  $$u^*u = p \quad \text{and} \quad uu^* = q.$$ 

- We write $p \leq q$ if there exists a partial isometry $u \in M$ such that
  
  $$u^*u = p \quad \text{and} \quad uu^* \leq q.$$ 

- A projection $p \neq 0$ is called minimal if for all projections $q \in M$,
  
  $$q \leq p \quad \Rightarrow \quad q = 0 \quad \text{or} \quad q = p.$$ 

- A projection $p \neq 0$ is called finite if for all projections $q \in M$,
  
  $$p \sim q \leq p \quad \Rightarrow \quad q = p.$$
Let $M \subset \mathcal{B}(\mathcal{H})$ be a factor. We say that $M$ is of type:

- $I$: if it contains a minimal projection;

\[ M \text{ is of type } I \iff M \cong \mathcal{B}(\mathcal{H}). \]

$\sim$ Type $I$ factors are classified by dimension.

- $II$: if it contains no minimal projection, but a finite projection;
  - Type $II_1$: if $1_{\mathcal{B}(\mathcal{H})}$ is finite (hence all projections are);
  - Type $II_\infty$: if $1_{\mathcal{B}(\mathcal{H})}$ is not finite, but some projections in $M$ are.

Any $II_\infty$ factor is of the form $M \otimes \mathcal{B}(\mathcal{H})$ with $M$ of type $II_1$ and $\dim \mathcal{H} = \infty$.

- $III$: if it contains no finite projection.
Refresher on Traces

### Trace of a matrix

The *trace* of a matrix $x = (x_{ij}) \in M_n(\mathbb{C})$ is $\text{Tr } x = \sum_i x_{ii}$. It satisfies

$$\text{Tr } ab = \text{Tr } ba$$

and

$$\text{Tr } p = \dim_{\mathbb{C}}(\text{range } p) = \dim_{\mathbb{C}} p \mathbb{C}^n$$

if $p$ is a projection.

### Exercise

Any linear map $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ such that

$$\varphi(ab) = \varphi(ba)$$

for all $a, b$ is a scalar multiple of the trace: $\varphi(x) = \frac{\varphi(I_n)}{n} \cdot \text{Tr } x$.

### Harder exercise

If $\mathcal{H}$ is infinite-dimensional, any linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ such that

$$\varphi(ab) = \varphi(ba)$$

for all $a, b$ is... 0.
Traces on von Neumann algebras

**Definition**
A linear functional \( \tau \) on a von Neumann algebra \( M \) is called:

- **positive** if \( \tau(x^*x) \geq 0 \) for all \( x \in M \)
- **faithful** if \( \tau(x^*x) = 0 \implies x = 0 \)
- **a state** if \( \tau(1) = 1 \)
- **tracial** if \( \tau(xy) = \tau(yx) \) for all \( x, y \in M \)

A tracial state is called a **trace**.

**Example:** consider the group factor of \( L(\Gamma) \subset \mathcal{B}(\ell^2(\Gamma)) \) associated with an icc group \( \Gamma \). Then

\[
\tau(x) = \langle x\delta_e, \delta_e \rangle
\]

extends to a faithful trace.

**Recall that:** \( g\delta_h = \delta_{gh} \) and \( \langle \delta_a, \delta_b \rangle = \delta_{a,b} \).
Characterization of $\mathcal{II}_1$ factors

Finite factors

Let $M$ be a factor. The following are equivalent:

- $M$ has a norm-continuous trace.
- 1 is a finite projection.
- $M$ is of type $I_n$ (with $n$ finite) or of type $\mathcal{II}_1$.

Under these conditions, $M$ is said finite.

**Remark:** 1 is a finite projection if and only if $u^*u = 1 \Rightarrow uu^* = 1$.

**Corollary**

Group factors $L(\Gamma)$ are of type $\mathcal{II}_1$.

The factor $L(S_\infty)$ is called *hyperfinite* and plays a special role in the theory. There are *many* more, constructed from group actions (Popa, Vaes,...)
Goals

• The Jones Index \(\leadsto\) *How do \(l_1\) factors sit within each other?*

• The Jones Tower

• Temperley-Lieb algebras, braids, knots, tangles...

*Left to right: Vaughan Jones and the HOMFLY crew, 1985*

J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W. B. R. Lickorish, and D. Yetter
Standard Form of a $II_1$ factor

Let $M$ be a factor with continuous and faithful trace $\tau$. The pairing
\[ \langle x, y \rangle = \tau(y^* x) \in \mathbb{C} \]
defines an inner product on $M$.

Let $L^2(M, \tau)$ be the associated completion of $M$. There is an embedding:
\[ M \rightarrow L^2(M) \]
\[ x \mapsto \hat{x} \quad \text{and} \quad 1 \mapsto \Omega \]

$M$ is represented on the Hilbert space $L^2(M)$ by considering:
\[ \pi_\tau(x)\hat{y} = \hat{xy} := x\hat{y} \]
and extending it to a morphism
\[ \pi_\tau : M \rightarrow \mathfrak{B}(L^2(M)) \].

This representation is called the **standard form** of $M$ and we have:
\[ \hat{x} = \hat{x1} = \hat{x1} = x\Omega \]
so that $\tau(x) = \langle x\Omega, \Omega \rangle$. 
Consider $M = M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ with normalized trace $\tau = \frac{1}{n} \text{Tr}$.

Then

$$L^2(M) = M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$$

with inner product

$$\langle x, y \rangle = \frac{\text{Tr} \ y^* x}{n}$$

and special vector

$$\Omega = I_n.$$

The standard representation is given by matrix multiplication:

$$\pi_\tau(x)y = xy.$$

**Note:** $L^2(M) \simeq \mathbb{C}^{n^2} \simeq \mathbb{C}^n \otimes \mathbb{C}^n$ and this representation has dimension $n^2$. 
Let $M$ be a $II_1$ factor.

A (left) $M$-module is a representation of $M$, that is, a Hilbert space $\mathcal{H}$ together with an action

$$M \times \mathcal{H} \to \mathcal{H}$$

$$(x, \xi) \mapsto x\xi$$

that is bilinear and satisfies

$$x(y\xi) = (xy)\xi \quad , \quad 1\xi = \xi \quad , \quad \langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle$$

for all $x, y \in M$ and $\xi, \eta \in \mathcal{H}$.

The action is also required to be continuous in the following sense: if $\{x_n\}_n$ is a bounded sequence in $M$, then

$$\tau(x_n^*x_n) \to 0 \quad \Rightarrow \quad ||x_n\xi|| \to 0$$

for any $\xi \in \mathcal{H}$.
Examples of $M$-modules

- The standard form: $L^2(M)$.
- Let $p$ be a projection in $M' \subset \mathcal{B}(L^2(M))$. Consider $\mathcal{H} = pL^2(M)$. Then,

$$xp\xi = px\xi$$

for any $x \in M$ and $\xi \in L^2(M)$. Denote by $L^2(M)p$ this $M$-module.

- Amplifications:

$$L^2(M) \oplus \ldots \oplus L^2(M)$$

- Infinite amplification:

$$\ell^2(L^2(M)) \simeq \ell^2(\mathbb{N}) \otimes L^2(M).$$

$$\ell^2(L^2(M)) = \{(\xi_n)_n : \xi_n \in L^2(M) \quad \text{with} \quad \sum_{n \in \mathbb{N}} \|\xi_n\|^2 < \infty\}. $$

**Goal:** quantitatively compare these examples with $L^2(M)$. 
Equivalent Representations

Let $M$ be a $\mathcal{II}_1$ factor and $\mathcal{H}_1$, $\mathcal{H}_2$ two $M$-modules.

**Definition**

We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *(unitarily) equivalent* and we write $\mathcal{H}_1 \simeq \mathcal{H}_2$ if there is a unitary

$$u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

such that

$$u(x\xi) = x(u\xi)$$

for all $x \in M$ and $\xi \in \mathcal{H}_1$.

In other words, there is a commutative diagram:
Theorem

Let $M$ be a $II_1$ factor and $\mathcal{H}$ a separable $M$-module. Then there is an isometry

$$v : \mathcal{H} \rightarrow L^2(M) \otimes \ell^2(\mathbb{N})$$

such that

$$vx = (x \otimes 1)v$$

for all $x \in M$. Furthermore,

$$vv^* \in (M \otimes 1)' \subset \mathcal{B}(L^2(M) \otimes \ell^2(\mathbb{N})).$$

Reformulation

Any representation of $M$ is equivalent to

$$p(L^2(M) \otimes \ell^2(\mathbb{N}))$$

for some projection $p = vv^*$ in $(M \otimes 1)'$. 
Any representation $\mathcal{H}$ of a II$_1$ factor $M$ is equivalent to $p(L^2(M) \otimes \ell^2(\mathbb{N}))$ for some projection $p = vv^*$ in $(M \otimes 1)'$.

**Idea:** the size of $\mathcal{H}$ is measured by the trace of $p$.

**Issue:** $(M \otimes 1)' = M' \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ is a II$_\infty$ factor! (No trace on $\mathcal{B}(\ell^2(\mathbb{N}))$...)

**Fix:** the trace

$$\text{tr} = \tau \otimes \text{Tr}$$

can be normalized in such a way that $\text{tr}(1 \otimes q) = 1$ for any rank 1 projection $q$ in $\mathcal{B}(\ell^2)(\mathbb{N})$.

**Definition**

The *$M$-dimension* of the $M$-module $\mathcal{H}$ is

$$\dim_M \mathcal{H} = \text{tr}(vv^*) \in [0, \infty].$$

Fortunately, $\text{tr}(vv^*)$ does not depend on a choice of the isometry $v$. 
Properties of the von Neumann Dimension

Let $M$ be a $II_1$ factor.

- $\dim_M \mathcal{H} \in [0, \infty]$ and all values occur.

- Representations of $M$ are classified by their $M$-dimension:
  \[ \dim_M \mathcal{H} = \dim_M \mathcal{K} \iff \mathcal{H} \simeq \mathcal{K}. \]

- Direct sums: if $J$ is countable then,
  \[ \dim_M \bigoplus_{j \in J} \mathcal{H}_j = \sum_{j \in J} \dim_M \mathcal{H}_j. \]

- If $p \in M'$ is a projection then:
  \[ \dim_M L^2(M)p = \tau(p). \]

- $M'$-dimension:
  \[ \dim_{M'} \mathcal{H} = \frac{1}{\dim_M \mathcal{H}}. \]
The Jones Index

Let \( N \subset M \) be \( II_1 \) factors.

Then \( N \) acts on \( L^2(M) \) and we can consider its \( N \)-dimension.

**Definition**

The *Jones Index* of the subfactor \( N \subset M \) is

\[
\]

**Proposition**

If \( N \subset M \) is represented on \( \mathcal{H} \) with \( \dim_N \mathcal{H} < \infty \), then

\[
[M : N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.
\]
Sketch of Proof

Assume $\dim_M \mathcal{H} \geq 1$. Note that $N \subset M$ implies $M' \subset N'$ and they are both $ll_1$ factors.

Let $p \in M'$ with

$$
\tau_{M'}(p) = \frac{1}{\dim_M \mathcal{H}}.
$$

Then

$$
\dim_M (\mathcal{H}p) = \tau_{M'}(p) \cdot \dim_M \mathcal{H} = 1
$$

and therefore

$$
\mathcal{H}p \simeq L^2(M).
$$

It follows that

$$
[M : N] = \dim_N L^2(M)
= \dim_N \mathcal{H}p
= \tau_{N'}(p) \cdot \dim_N \mathcal{H} = \tau_{M'}(p) \cdot \dim_N \mathcal{H}
= \frac{1}{\dim_M \mathcal{H}} \dim_N \mathcal{H}
\square
$$
Example

Let $N$ be a $II_1$ factor. For $k \geq 1$ integer, consider

$$M = M_k(N) \cong N \otimes M_k(\mathbb{C}).$$

Then $N$ is a subfactor of $M$, via the embedding

$$N \ni x \mapsto x \otimes 1 = \begin{bmatrix}
    x & 0 & \cdots & 0 \\
    0 & x & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & x & 0
\end{bmatrix} \in M_k(N)$$

Moreover,

$$L^2(M) = L^2(N \otimes M_k(\mathbb{C})) \cong L^2(N) \otimes M_k(\mathbb{C}) \cong \bigoplus_{j=1}^{k^2} L^2(N),$$

so that

$$[M : N] = \dim_N L^2(M) = \sum_{j=1}^{k^2} \dim_N L^2(N) = k^2.$$
The Jones Index Theorem

~ Beside 1, 4, 9, ... what possible values can $[M : N]$ take?

**Theorem (Jones, 1983)**

Let $N \subset M$ be a $II_1$ subfactor. The possible values of $[M : N]$ are given by:

$$\left\{ 4 \cos^2\left(\frac{\pi}{n+2}\right) : n \geq 1 \right\} \cup [4, \infty).$$

The proof relies on studying indices along the *Jones Tower* and relates in profound ways the algebra of projections and Temperly-Lieb algebras.

"Corollary"

A new invariant for knots: the Jones polynomial (and its descendants).
'God may or may not play dice but She sure loves a von Neumann algebra.'

Vaughan Jones
1952 - 2020