Von Neumann Algebras, Subfactor and Knots

II. Subfactors

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GAG seminar

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1. Summary of Last Week

2. Factors of Type II_1

3. Representation Theory of II_1 Factors and the Jones Index

Operator Algebras

If $\mathcal{H}$ is a Hilbert space.

Denote by $\mathfrak{B}(\mathcal{H})$ the space of bounded operators from $\mathcal{H}$ to itself.

Theorem

If $T \in \mathfrak{B}(\mathcal{H})$, there exists a unique operator T^* such that:

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$$

for all ξ, η in $\mathcal{H}$.

We are interested in subalgebras of $\mathfrak{B}(\mathcal{H})$ that are stable under $T \mapsto T^*$.

C*-algebras

Definition

A C*-algebra is a closed subalgebra $A \subset \mathfrak{B}(\mathcal{H})$ that is stable under adjoint.

I. Gelfand (1913 - 2009)

Examples:

- $M_n(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^n)$
- $\mathfrak{B}(\mathcal{H})$: bounded operators on $\mathcal{H}$
- $\mathfrak{K}(\mathcal{H})$: compact operators on $\mathcal{H}$
- A commutative $\Rightarrow A \simeq C_0(X)$.

Commutants

The *commutant* of a subset $A \subset \mathfrak{B}(\mathcal{H})$ is

$$A' := \Big\{ y \in \mathfrak{B}(\mathcal{H})\,,\ xy = yx \quad \text{for all } x \in A \Big\}.$$

The *bicommutant* of A is

$$A'' = (A')'.$$

Remark: $A \subset A''$ for any A.

Example: let $A = M_n(\mathbb{C}) \simeq \mathfrak{B}(\mathbb{C}^n)$. Then

$$A' = \mathbb{C}.I_n \qquad \text{and} \qquad A'' = M_n(\mathbb{C}) = A$$

Definition

A *von Neumann algebra* is a $*$-subalgebra M of $\mathfrak{B}(\mathcal{H})$ such that $M = M''$.

John von Neumann
(1903 - 1957)

Examples:

- $M_n(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^n)$
- $\mathfrak{B}(\mathcal{H})$: bounded operators on $\mathcal{H}$
- M commutative $\Rightarrow M \simeq L^\infty(X, \mu)$.

Factors

Definition

A von Neumann algebra $M \subset \mathfrak{B}(\mathcal{H})$ is called a *factor* if

$$M \cap M' = \mathbb{C}.1_{\mathfrak{B}(\mathcal{H})}$$

Structure Theorem

Any separable von Neumann algebra M is isomorphic to a direct integral of factors:

$$M \simeq \int_{\mathcal{T}}^{\oplus} M_t \, d\mu(t)$$

Group von Neumann Algebras

Let Γ be a discrete group with group algebra

$$\mathbb{C}\Gamma = \Big\{ \sum_{\gamma \in \Gamma} \alpha_\gamma . \gamma \, , \; \alpha_\gamma \in \mathbb{C} \quad \text{and} \quad \alpha_\gamma \neq 0 \quad \text{only for finitely many } \gamma \Big\}.$$

Then $\mathbb{C}\Gamma$ acts on the Hilbert space $\ell^2(\Gamma)$ by left multiplication:

$$\mathbb{C}\Gamma \subset \mathfrak{B}\big(\ell^2(\Gamma)\big)$$

and the *von Neumann algebra of* Γ is:

$$L(\Gamma) := \overline{\mathbb{C}\Gamma}^{SOT} = \overline{\mathbb{C}\Gamma}^{WOT} = \mathbb{C}\Gamma''$$

Theorem

$L(\Gamma)$ is a factor $\quad \Leftrightarrow \quad$ Γ has infinite conjugacy classes.

The Free Group Isomorphism Problem

If $m \neq n$, is it the case that $L(\mathbb{F}_m) \not\cong L(\mathbb{F}_n)$? *Still open...*

Algebra of Projections (Murray - von Neumann, 1936)

A *projection* in a von Neumann algebra M is an element $p \in M$ such that:

$$p^* = p = p^2.$$

We write $p \leq q$ if $p\mathcal{H} \subset q\mathcal{H}$.

- Two projection $p, q \in M$ are said *equivalent* ($p \sim q$) if there is a partial isometry $u \in M$ such that
$$u^* u = p \quad \text{and} \quad uu^* = q.$$
- We write $p \precsim q$ if there exists a partial isometry $u \in M$ such that
$$u^* u = p \quad \text{and} \quad uu^* \leq q.$$
- A projection $p \neq 0$ is called *minimal* if for all projections $q \in M$,
$$q \leq p \quad \Rightarrow \quad q = 0 \quad \text{or} \quad q = p.$$
- A projection $p \neq 0$ is called *finite* if for all projections $q \in M$,
$$p \sim q \leq p \quad \Rightarrow \quad q = p.$$

Murray - von Neumann Classification (1936 - 1943)

Let $M \subset \mathfrak{B}(\mathcal{H})$ be a factor. We say that M is of type:

- *I*: if it contains a minimal projection;

$$M \text{ is of type } I \quad \Leftrightarrow \quad M \simeq \mathfrak{B}(\mathcal{H}).$$

$\rightsquigarrow$ Type *I* factors are classified by dimension.

- *II*: if it contains no minimal projection, but a finite projection;
  - Type II_1: if $1_{\mathfrak{B}(\mathcal{H})}$ is finite (hence all projections are);
  - Type II_∞: if $1_{\mathfrak{B}(\mathcal{H})}$ is not finite, but some projections in M are.

Any II_∞ factor is of the form $M \otimes \mathfrak{B}(\mathcal{H})$ with M of type II_1 and $\dim \mathcal{H} = \infty$.

- *III*: if it contains no finite projection.

Refresher on Traces

Trace of a matrix

The *trace* of a matrix $x = (x_{ij}) \in M_n(\mathbb{C})$ is $\operatorname{Tr} x = \sum_i x_{ii}$. It satisfies

$$\operatorname{Tr} ab = \operatorname{Tr} ba \qquad \text{and} \qquad \operatorname{Tr} p = \dim_{\mathbb{C}}(\operatorname{range} p) = \dim_{\mathbb{C}} p\mathbb{C}^n$$

if p is a projection.

Exercise

Any linear map $\varphi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ such that

$$\varphi(ab) = \varphi(ba)$$

for all a, b is a scalar multiple of the trace: $\varphi(x) = \frac{\varphi(I_n)}{n} . \operatorname{Tr} x$.

Harder exercise

If $\mathcal{H}$ is infinite-dimensional, any linear map $\varphi : \mathfrak{B}(\mathcal{H}) \longrightarrow \mathbb{C}$ such that

$$\varphi(ab) = \varphi(ba)$$

for all a, b is... 0.

Traces on von Neumann algebras

Definition

A linear functional τ on a von Neumann algebra M is called:

- *positive* if $\tau(x^*x) \geq 0$ for all $x \in M$
- *faithful* if $\tau(x^*x) = 0 \quad \Rightarrow \quad x = 0$
- *a state* if $\tau(1) = 1$
- *tracial* if $\tau(xy) = \tau(yx)$ for all $x, y \in M$

A tracial state is called a *trace*.

Example: consider the group factor of $L(\Gamma) \subset \mathfrak{B}(\ell^2(\Gamma))$ associated with an icc group Γ. Then

$$\tau(x) = \langle x\delta_e, \delta_e \rangle$$

extends to a faithful trace.

Recall that: $g\delta_h = \delta_{gh}$ *and* $\langle \delta_a, \delta_b \rangle = \delta_{a,b}$.

Characterization of II_1 factors

Finite factors

Let M be a factor. The following are equivalent:

- M has a norm-continuous trace.
- 1 is a finite projection.
- M is of type I_n (with n finite) or of type II_1.

Under these conditions, M is said *finite*.

Remark: 1 is a finite projection if and only if $u^*u = 1 \Rightarrow uu^* = 1$.

Corollary

Group factors $L(\Gamma)$ are of type II_1.

The factor $L(S_\infty)$ is called *hyperfinite* and plays a special role in the theory. There are *many* more, constructed from group actions (Popa, Vaes,...)

Goals

- The Jones Index $\rightsquigarrow$ *How do II_1 factors sit within each other?*
- The Jones Tower
- Temperley-Lieb algebras, braids, knots, tangles...

Left to right: Vaughan Jones and the HOMFLY crew, 1985
J. **H**oste, A. **O**cneanu, K. **M**illett, P. **F**reyd, W. B. R. **L**ickorish, and D. **Y**etter

Standard Form of a II_1 factor

Let M be a factor with continuous and faithful trace τ. The pairing

$$\langle x, y \rangle = \tau(y^* x) \in \mathbb{C}$$

defines an inner product on M.
Let $L^2(M, \tau)$ be the associated completion of M. There is an embedding:

$$\begin{array}{ccc} M & \longrightarrow & L^2(M) \\ x & \longmapsto & \hat{x} \\ 1 & \longmapsto & \Omega \end{array} .$$

M is represented on the Hilbert space $L^2(M)$ by considering:

$$\pi_\tau(x)\hat{y} = \widehat{xy} := x\hat{y}$$

and extending it to a morphism

$$\pi_\tau : M \longrightarrow \mathfrak{B}(L^2(M)).$$

This representation is called the *standard form* of M and we have:

$$\hat{x} = \widehat{x1} = x\hat{1} = x\Omega$$

so that $\tau(x) = \langle x\Omega, \Omega \rangle$.

Example

Consider $M = M_n(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^n)$ with normalized trace $\tau = \frac{1}{n}\operatorname{Tr}$.

Then

$$L^2(M) = M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$$

with inner product

$$\langle x, y \rangle = \frac{\operatorname{Tr} y^* x}{n}$$

and special vector

$$\Omega = I_n.$$

The standard representation is given by matrix multiplication:

$$\pi_\tau(x) y = xy.$$

Note: $L^2(M) \simeq \mathbb{C}^{n^2} \simeq \mathbb{C}^n \otimes \mathbb{C}^n$ and this representation has dimension n^2.

Modules

Let M be a II_1 factor.

A (left) *M-module* is a representation of M, that is, a Hilbert space $\mathcal{H}$ together with an action

$$\begin{aligned} M \times \mathcal{H} &\longrightarrow \mathcal{H} \\ (x, \xi) &\longmapsto x\xi \end{aligned}$$

that is bilinear and satisfies

$$x(y\xi) = (xy)\xi \quad , \quad 1\xi = \xi \quad , \quad \langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle$$

for all $x, y \in M$ and $\xi, \eta \in \mathcal{H}$.

The action is also required to be *continuous* in the following sense: if $\{x_n\}_n$ is a bounded sequence in M, then

$$\tau(x_n^* x_n) \to 0 \quad \Rightarrow \quad \|x_n \xi\| \to 0$$

for any $\xi \in \mathcal{H}$.

Examples of M-modules

- The standard form: $L^2(M)$.
- Let p be a projection in $M' \subset \mathfrak{B}(L^2(M))$. Consider $\mathcal{H} = pL^2(M)$. Then,
$$xp\xi = px\xi$$
for any $x \in M$ and $\xi \in L^2(M)$. Denote by $L^2(M)p$ this M-module.
- Amplifications:
$$L^2(M) \oplus \ldots \oplus L^2(M)$$
- Infinite amplification:
$$\ell^2(L^2(M)) \simeq \ell^2(\mathbb{N}) \otimes L^2(M).$$

$$\ell^2(L^2(M)) = \Big\{(\xi_n)_n \ : \ \xi_n \in L^2(M) \quad \text{with} \quad \sum_{n\in\mathbb{N}} \|\xi_n\|^2 < \infty\Big\}.$$

Goal: quantitatively compare these examples with $L^2(M)$.

Equivalent Representations

Let M be a II_1 factor and $\mathcal{H}_1, \mathcal{H}_2$ two M-modules.

Definition

We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *(unitarily) equivalent* and we write $\mathcal{H}_1 \simeq \mathcal{H}_2$ if there is a unitary

$$u : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

such that

$$u(x\xi) = x(u\xi)$$

for all $x \in M$ and $\xi \in \mathcal{H}_1$.

In other words, there is a commutative diagram:

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "u"] \arrow[d, "x"] & \mathcal{H}_2 \arrow[d, "x"] \\ \mathcal{H}_1 \arrow[r, "u"] & \mathcal{H}_2 \end{tikzcd}$$

Representation Theory of II_1 Factors (all of it!)

Theorem

Let M be a II_1 factor and $\mathcal{H}$ a separable M-module. Then there is an isometry

$$v : \mathcal{H} \longrightarrow L^2(M) \otimes \ell^2(\mathbb{N})$$

such that

$$vx = (x \otimes 1)v$$

for all $x \in M$. Furthermore,

$$vv^* \in (M \otimes 1)' \subset \mathfrak{B}\big(L^2(M) \otimes \ell^2(\mathbb{N})\big).$$

Reformulation

Any representation of M is equivalent to

$$p\big(L^2(M) \otimes \ell^2(\mathbb{N})\big)$$

for some projection $p = vv^*$ in $(M \otimes 1)'$.

Von Neumann Dimension

Any representation $\mathcal{H}$ of a II_1 factor M is equivalent to $p\big(L^2(M) \otimes \ell^2(\mathbb{N})\big)$ for some projection $p = vv^*$ in $(M \otimes 1)'$.

Idea: the size of $\mathcal{H}$ is measured by the trace of p.

Issue: $(M \otimes 1)' = M' \otimes \mathfrak{B}(\ell^2(\mathbb{N}))$ is a II_∞ factor! (No trace on $\mathfrak{B}(\ell^2(\mathbb{N}))$...)

Fix: the trace

$$\mathrm{tr} = \tau \otimes \mathrm{Tr}$$

can be normalized in such a way that $\mathrm{tr}(1 \otimes q) = 1$ for any rank 1 projection q in $\mathfrak{B}(\ell^2)(\mathbb{N})$.

Definition

The *M-dimension* of the M-module $\mathcal{H}$ is

$$\dim_M \mathcal{H} = \mathrm{tr}(vv^*) \in [0, \infty].$$

Fortunately, $\mathrm{tr}(vv^*)$ does not depend on a choice of the isometry v.

Properties of the von Neumann Dimension

Let M be a II_1 factor.

- $\dim_M \mathcal{H} \in [0, \infty]$ and all values occur.
- Representations of M are classified by their M-dimension:
$$\dim_M \mathcal{H} = \dim_M \mathcal{K} \quad \Leftrightarrow \quad \mathcal{H} \simeq \mathcal{K}.$$
- Direct sums: if J is countable then,
$$\dim_M \bigoplus_{j \in J} \mathcal{H}_j = \sum_{j \in J} \dim_M \mathcal{H}_j.$$
- If $p \in M'$ is a projection then:
$$\dim_M L^2(M)p = \tau(p).$$
- M'-dimension:
$$\dim_{M'} \mathcal{H} = \frac{1}{\dim_M \mathcal{H}}.$$

The Jones Index

Let $N \subset M$ be II_1 factors.

Then N acts on $L^2(M)$ and we can consider its N-dimension.

Definition

The *Jones Index* of the subfactor $N \subset M$ is

$$[M : N] := \dim_N L^2(M).$$

Proposition

If $N \subset M$ is represented on $\mathcal{H}$ with $\dim_N \mathcal{H} < \infty$, then

$$[M : N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.$$

Sketch of Proof

Assume $\dim_M \mathcal{H} \geq 1$. Note that $N \subset M$ implies $M' \subset N'$ and they are both II_1 factors.
Let $p \in M'$ with

$$\tau_{M'}(p) = \frac{1}{\dim_M \mathcal{H}}.$$

Then

$$\dim_M(\mathcal{H}p) = \tau_{M'}(p) \cdot \dim_M \mathcal{H} = 1$$

and therefore

$$\mathcal{H}p \simeq L^2(M).$$

It follows that

$$\begin{aligned} [M:N] &= \dim_N L^2(M) \\ &= \dim_N \mathcal{H}p \\ &= \tau_{N'}(p) \cdot \dim_N \mathcal{H} = \tau_{M'}(p) \cdot \dim_N \mathcal{H} \\ &= \frac{1}{\dim_M \mathcal{H}} \dim_N \mathcal{H} \end{aligned}$$

□

Example

Let N be a II_1 factor. For $k \geq 1$ integer, consider

$$M = M_k(N) \simeq N \otimes M_k(\mathbb{C}).$$

Then N is a subfactor of M, via the embedding

$$N \ni x \longmapsto x \otimes 1 = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & & \vdots \\ \vdots & & x & 0 \\ 0 & \cdots & 0 & x \end{bmatrix} \in M_k(N)$$

Moreover,

$$L^2(M) = L^2\Big(N \otimes M_k(\mathbb{C})\Big) \simeq L^2(N) \otimes M_k(\mathbb{C}) \simeq \bigoplus_{j=1}^{k^2} L^2(N),$$

so that

$$[M : N] = \dim_N L^2(M) = \sum_{j=1}^{k^2} \dim_N L^2(N) = k^2.$$

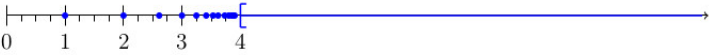
The Jones Index Theorem

$\rightsquigarrow$ Beside $1, 4, 9, \ldots$ what possible values can $[M : N]$ take?

Theorem (Jones, 1983)

Let $N \subset M$ be a II_1 subfactor. The possible values of $[M : N]$ are given by:

$$\left\{4\cos^2\left(\frac{\pi}{n+2}\right) \;:\; n \geq 1\right\} \cup [4, \infty).$$



The proof relies on studying indices along the *Jones Tower* and relates in profound ways the algebra of projections and Temperly-Lieb algebras.

"Corollary"

A new invariant for knots: the Jones polynomial (and its descendants).

‘God may or may not play dice but She sure loves a von Neumann algebra.’

Vaughan Jones
1952 - 2020