Von Neumann Algebras, Subfactor and Knots

I. Von Neumann Algebras, Factors

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GAG seminar

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What is an observable quantity?

If you ask Wikipedia:

Quantum mechanics [edit]

In quantum physics, observables manifest as linear operators on a Hilbert space representing the state space of quantum states.

$\rightsquigarrow$ We should study linear operators on Hilbert spaces.

Hilbert spaces

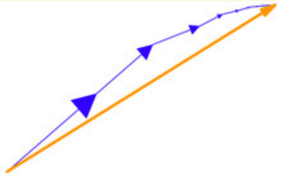
A *Hilbert space* is a complex vector space $\mathcal{H}$ equipped with an *inner product*

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$$

that is *complete* for the associated norm

$$\|\xi\| := \sqrt{\langle \xi, \xi \rangle}.$$

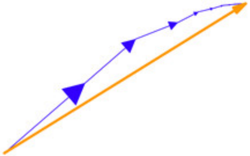
From Wikipedia:



'Completeness means that if a particle moves along the broken path (in blue) travelling a finite total distance, then the particle has a well-defined net displacement (in orange).'

Completeness vs. completeness

A metric space is said *complete* if all Cauchy sequences converge.



Theorem

A normed linear space $(E, \|\cdot\|)$ is complete if and only if absolutely convergent series are convergent, that is:

$$\sum_{n\geq 0} \|\xi_n\| < +\infty \qquad \Rightarrow \qquad \sum_{n\geq 0} \xi_n \text{ converges in } E.$$

'Completeness means that if a particle moves along the broken path travelling a finite total distance, then the particle has a well-defined net displacement.'

Bounded operators

We will focus on bounded operators on $\mathcal{H}$, that is, continuous linear maps

$$T : \mathcal{H} \longrightarrow \mathcal{H}.$$

Theorem

Let $T : E \longrightarrow F$ be a linear map between normed linear spaces. The following are equivalent:

- T is continuous on E.
- T is bounded on the unit ball of E.
- T is *Lipschitz*:

$$\|Tx\| \leq K\|x\| \quad \text{for all } x \in E.$$

If T is continuous, we define its *operator norm* by:

$$\|T\|_{op} = \sup_{\|x\|\leq 1} \|Tx\| = \inf \Big\{K \,,\, K \text{ is a Lipschitz constant for } T\Big\}.$$

Adjoints

If $\mathcal{H}$ is a Hilbert space, denote by $\mathfrak{B}(\mathcal{H})$ the space of bounded operators from $\mathcal{H}$ to itself.The operator norm makes it a *Banach algebra*.

Theorem

If $T \in \mathfrak{B}(\mathcal{H})$, there exists a unique operator T^* such that:

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$$

for all ξ, η in $\mathcal{H}$.

Properties of the adjoint:

- If $\mathcal{H}$ is finite-dimensional, $\mathrm{mat}(T^*) = \overline{{}^t\,\mathrm{mat}(T)}$
- Adjunction is an isometry:

$$\|T^*\|_{op} = \|T\|_{op}$$

C*-identity

For every T in $\mathfrak{B}(\mathcal{H})$,

$$\|T^*T\|_{op} = \|T\|_{op}^2$$

(Concrete) C*-algebras

Definition

A C*-algebra is a closed subalgebra of $\mathfrak{B}(\mathcal{H})$ that is stable under adjoint.

I. Gelfand (1913 - 2009)

Examples:

- $\mathbb{C}$, with $z^* = \bar{z}$ and $\|z\| = |z|$
- $M_n(\mathbb{C}) = \mathfrak{B}(\mathbb{C}^n)$
- $\mathfrak{B}(\mathcal{H})$: bounded operators on $\mathcal{H}$
- $\mathfrak{K}(\mathcal{H})$: compact operators on $\mathcal{H}$

Commutative C*-algebras

Theorem (Gelfand-Naimark, 1943)

If A is a commutative C*-algebra, there exists a locally compact Hausdorff space X (the *spectrum* of A) such that

$$A \simeq C_0(X).$$

- The C*-algebra structure on $C_0(X)$ is given by:

$$\|f\| = \sup_{x \in X} |f(x)| \qquad \text{and} \qquad f^*(x) = \overline{f(x)}.$$

- $C_0(X)$ is unital *if and only if* X is compact.

C*-algebras = *noncommutative topology*

John von Neumann (1903 - 1957)

Commutants

The *commutant* of a subset $A \subset \mathfrak{B}(\mathcal{H})$ is

$$A' := \Big\{ y \in \mathfrak{B}(\mathcal{H})\,,\; xy = yx \quad \text{for all } x \in A \Big\}.$$

The *bicommutant* of A is

$$A'' = (A')'.$$

Remark: $A \subset A''$ for any A.

Example: let $A = M_n(\mathbb{C}) \simeq \mathfrak{B}(\mathbb{C}^n)$. Then

$$A' = \mathbb{C}.I_n \qquad \text{and} \qquad A'' = M_n(\mathbb{C}) = A$$

Question

Say A is stable under algebraic operations and taking adjoints. What is the difference between A and A''?

The Bicommutant Theorem

$\mathfrak{B}(\mathcal{H})$ carries other useful topologies than the one given by $\|\cdot\|_{op}$: the *strong operator topology* (SOT) and the *weak operator topology* (WOT).

$$\text{WOT} \quad \prec \quad \text{SOT} \quad \prec \quad \text{norm topology}$$

In particular, for any subset X of $\mathfrak{B}(\mathcal{H})$,

$$\overline{X}^{\|\cdot\|_{op}} \subset \overline{X}^{SOT} \subset \overline{X}^{WOT}$$

Von Neumann's Bicommutant Theorem

Let M be a unital $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$. Then

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''.$$

W*- and von Neumann Algebras

Recall:

Definition

A C*-algebra is a closed $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$.

Similarly,

Definition

A W*-algebra is a weakly closed $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$.

Equivalently, by the Bicommutant Theorem:

Definition

A *von Neumann algebra* is a $*$-subalgebra M of $\mathfrak{B}(\mathcal{H})$ such that $M = M''$.

Commutative von Neumann Algebras

If X is a space carrying a σ-finite measure μ, and $f \in L^\infty(X, \mu)$, then

$$M_f : \varphi \longmapsto f.\varphi$$

is a bounded operator on the Hilbert space $L^2(X, \mu)$, with $\|M_f\|_{op} = \|f\|_\infty$.

This is the only possibility!

Theorem

A commutative von Neumann algebra of operators on a separable Hilbert space is $*$-isomorphic to exactly one of the following:

- $\ell^\infty(\{1, 2, \ldots, n\})$ with $n \geq 1$
- $\ell^\infty(\mathbb{N})$
- $L^\infty([0, 1])$
- $L^\infty([0, 1] \cup \{1, 2, \ldots, n\})$ with $n \geq 1$
- $L^\infty([0, 1] \cup \mathbb{N})$

Two-line Summary of Noncommutative Analysis

C*-algebras = *noncommutative topology*

von Neumann algebras = *noncommutative measure theory*

$\rightsquigarrow$ What are the building blocks of the theory?

Factors

Definition

A von Neumann algebra $M \subset \mathfrak{B}(\mathcal{H})$ is called a *factor* if

$$M \cap M' = \mathbb{C}.1_{\mathfrak{B}(\mathcal{H})}$$

Remark: $M \cap M' = Z(M)$ does not depend on the Hilbert space $\mathcal{H}$.

Structure Theorem

Any separable von Neumann algebra M is isomorphic to a direct integral of factors: there exists a measurable space $\mathcal{T}$ with a σ-finite measure μ and a family $\{M_t\}_{t\in\mathcal{T}}$ of factors such that:

$$M \simeq \int_{\mathcal{T}}^{\oplus} M_t \, d\mu(t)$$

Remark: $\mathbb{C}$ is the only commutative factor.

Group von Neumann Algebras

Let Γ be a discrete group with group algebra

$$\mathbb{C}\Gamma = \Big\{ \sum_{\gamma\in\Gamma} \alpha_\gamma.\gamma \,,\ \alpha_\gamma \in \mathbb{C} \quad \text{and} \quad \alpha_\gamma \neq 0 \quad \text{only for finitely many } \gamma \Big\}.$$

Then $\mathbb{C}\Gamma$ acts on the Hilbert space $\ell^2(\Gamma)$ by left multiplication:

$$\mathbb{C}\Gamma \subset \mathfrak{B}\big(\ell^2(\Gamma)\big)$$

and the *von Neumann algebra of* Γ is:

$$L(\Gamma) := \overline{\mathbb{C}\Gamma}^{SOT} = \overline{\mathbb{C}\Gamma}^{WOT} = \mathbb{C}\Gamma''$$

Theorem

$L(\Gamma)$ is a factor $\quad \Leftrightarrow \quad$ Γ has infinite conjugacy classes.

The Free Group Isomorphism Problem

If $m \neq n$, is it the case that $L(\mathbb{F}_m) \neq L(\mathbb{F}_n)$? *Still open...*

(How) can factors be classified?

- Yes and no...
- They can be organized in three generic families using
  - *projections* and *traces* (Murray - von Neumann, 1930's)
  - the *flow of weights* (Connes, 1970's).
- Useful invariants have been constructed, but there is no hope of classification (except for the first family)...

$\rightsquigarrow$ It's a good thing!

Partial Isometries

Definition

An element u in $\mathfrak{B}(\mathcal{H})$ is called a *partial isometry* if

$$\|u\xi\| = \|\xi\|$$

for all $\xi \in (\ker u)^{\perp}$.

Example: consider the shift operators on $\mathcal{H} = \ell^2(\mathbb{N})$:

$$R : (x_1, x_2, \ldots) \longmapsto (0, x_1, x_2, \ldots)$$

$$L : (x_1, x_2, \ldots) \longmapsto (x_2, x_3, \ldots)$$

They are adjoint to each other and satisfy:

$$R^* R = LR = Id_{\ell^2(\mathbb{N})}$$

$$RR^* = RL = \text{projection onto } (0, *, *, *, \ldots) = (\ker L)^{\perp}$$

Algebra of Projections (Murray - von Neumann, 1936)

A *projection* in a von Neumann algebra M is an element $p \in M$ such that:

$$p^* = p = p^2.$$

We write $p \leq q$ if $p\mathcal{H} \subset q\mathcal{H}$.

- Two projection $p, q \in M$ are said *equivalent* ($p \sim q$) if there is a partial isometry $u \in M$ such that

$$u^*u = p \quad \text{and} \quad uu^* = q.$$

- We write $p \precsim q$ if there exists a partial isometry $u \in M$ such that

$$u^*u = p \quad \text{and} \quad uu^* \leq q.$$

- A projection $p \neq 0$ is called *minimal* if for all projections $q \in M$,

$$q \leq p \quad \Rightarrow \quad q = 0 \;\text{ or }\; q = p.$$

- A projection $p \neq 0$ is called *finite* if for all projections $q \in M$,

$$p \sim q \leq p \quad \Rightarrow \quad q = p.$$

Murray - von Neumann Classification (1936 - 1943)

Let $M \subset \mathfrak{B}(\mathcal{H})$ be a factor. We say that M is of type:

- *I*: if it contains a minimal projection;

$$M \text{ is of type } I \quad \Leftrightarrow \quad M \simeq \mathfrak{B}(\mathcal{H}).$$

$\rightsquigarrow$ Type *I* factors are classified by dimension.

- *II*: if it contains no minimal projection, but a finite projection;
  - Type II_1: if $1_{\mathfrak{B}(\mathcal{H})}$ is finite (hence all projections are);
  - Type II_∞: if $1_{\mathfrak{B}(\mathcal{H})}$ is not finite, but some projections in M are.

Any II_∞ factor is of the form $M \otimes \mathfrak{B}(\mathcal{H})$ with M of type II_1 and $\dim \mathcal{H} = \infty$.

- *III*: if it contains no finite projection.
  - Type III_λ with $0 \leq \lambda \leq 1$, from the Connes spectrum (1970's)
  - Every *III* factor is of the form $M \rtimes \mathbb{R}$ with M of type II_∞ (Tomita-Takesaki)