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The Minimum Number of Multiplicity 1 Eigenvalues among Real Symmetric Matrices whose Graph is a Tree

Wenxuan "Olivia" Ding
Advisor: Dr. Charles R. Johnson

College of William & Mary

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Introduction

Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix. The **graph** of A, denoted $G(A)$, is the simple undirected graph on n vertices with an edge between i and j iff the entry $a_{ij} \neq 0$ (no restriction on the diagonal entries). Given a graph G, we define $\mathcal{S}(G)$ to be the set of all real symmetric matrices A such that $G(A) = G$.

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$$T = \text{①}-\text{②}-\text{③}-\text{④}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix}$$

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For example, when $n = 15$, and the eigenvalues are

$$-3, -1, -1, 2, 4, 4, 4, 5, 5, 6, 8, 8, 10, 11, 25$$

An ordered multiplicity list is $(1, 2, 1, 3, 2, 1, 2, 1, 1, 1)$, while an unordered multiplicity list is $(3, 2, 2, 2, 1, 1, 1, 1, 1, 1)$.

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The **catalog** of a tree, denoted $\mathcal{L}(T)$, is the collection of all multiplicity lists that occur among the matrices in $\mathcal{S}(T)$.
We define $U(T)$ to be the minimum number of 1's among the lists in $\mathcal{L}(T)$.

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There has been much interest, and progress on determining $\mathcal{L}(T)$ for each tree T. The maximum multiplicity, $M(T)$, is the path cover number $P(T)$ [7], and the minimum number of distinct eigenvalues is at least the diameter $d(T)$ [7]. Similarly, precise information about $U(T)$ would further narrow the possibilities for the catalog $\mathcal{L}(T)$.

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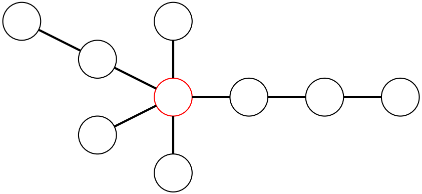

Background

A tree is a **linear tree** if all its HDV's lie on a single induced path. A linear tree with k HDV's is called **k-linear**. And a linear tree can be viewed as the composition of g-stars and connecting paths, i.e. $T = L(T_1, s_1, \ldots, s_{k-1}, T_k)$ [2].

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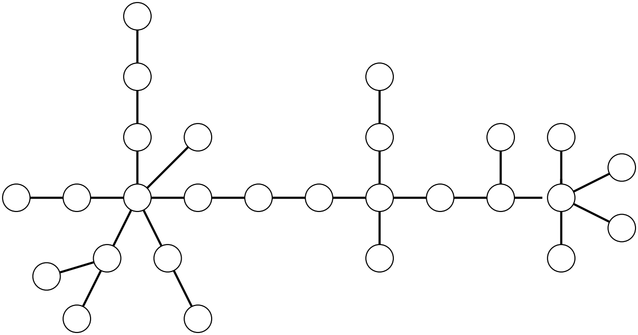
For example, the following tree is a 4-linear tree.



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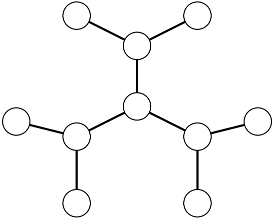
The smallest **nonlinear** tree is on 10 vertices, and by 25 vertices, half of the trees are linear.



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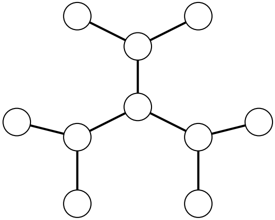

The **diameter** is defined to be the length of the longest induced path in T, measured in vertices. For example, the diameter for this smallest nonlinear tree is 5.

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Auxiliary results for g-stars and their upward multiplicity lists:

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Therefore, for a g-star T, we have the following lemma:

Lemma

[1] *Let T be a g-star with the central vertex v. If $A \in \mathcal{S}(T)$ and λ is an eigenvalue of $A(v)$, then $m_{A(v)}(\lambda) = m_A(\lambda) + 1$.*

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From this lemma, we can say that the upward eigenvalues of A (including those with multiplicity $\hat{0}$) are exactly the eigenvalues of $A(v)$. The **complete upward multiplicity lists** of a g-star have the form $(1, \hat{q}_1, 1, \hat{q}_2, 1, \cdots, \hat{q}_r, 1)$, in which r upward multiplicities are "bookended" by $r + 1$ non-upward 1's.

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Linear superposition principle (LSP):

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Theorem

[2] [5] *Let $T_1, \cdots, T_k$ be g-stars and $s_1, \cdots, s_{k-1}$ be nonnegative integers. Given $\hat{b}_i$, a complete upward multiplicity list for T_i (relative to the central vertex), $i = 1, \cdots, k$, and $\hat{c}_j$, a list of s non-upward 1's, $j = 1, \cdots, k-1$, construct augmented lists b_i^+, $i = 1, \cdots, k$ and c_j^+ subject to the following:*

1. *all b_i^+ and c_j^+ are the same length;*
2. *each b_i^+ and c_j^+ is obtained from $\hat{b}_i$ ($\hat{c}_j$) by inserting nonupward 0s;*
3. *for each l, the l^{th} element of the augmented lists, denoted $b_{i,l}^+$ and $c_{j,l}^+$, are not all nonupward 0s; and*
4. *for each l, arranging the $b_{i,l}^+$'s and $c_{j,l}^+$'s in the order $b_{1,l}^+, c_{1,l}^+, b_{2,l}^+ c_{2,l}^+, \ldots, b_{k,l}^+$, there is at least one upward multiplicity between any two non-upward ones.*

Then $\sum_{i=1}^{k} b_i^+ + \sum_{j=1}^{k-1} c_j^+$,where the addition is termwise, is a multiplicity list for $LT(T_1, s_1, \ldots, s_{k-1}, T_k)$ generated by the LSP. For any k -linear tree $T = L(T_1, s_1, T_2, s_2, \ldots, s_{k-1}, T_k)$, $\mathcal{L}_o(T)$ is equal to the set of all candidate multiplicity lists generated by the LSP for T.

Background

Example

Let $T = L(T_1, 2, T_2)$ and let $\hat{b} = (1, \hat{2}, 1)$ and $\hat{c} = (1, \hat{1}, 1, \hat{1}, 1, \hat{1}, 1)$ be upward multiplicity lists of T_1 and T_2, respectively. The following are two ways superimposing to get ordered multiplicity lists for T:

$$\begin{array}{cccccccc} 0 & 1 & \hat{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & \hat{1} & 1 & \hat{1} & 1 & \hat{1} & 1 & 0 \\ \hline 1 & 2 & 3 & 2 & 1 & 2 & 1 & 1 \end{array}$$

and

$$\begin{array}{cccccccccc} 0 & 0 & 1 & 0 & \hat{2} & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \hat{1} & 0 & 1 & \hat{1} & 1 & \hat{1} & 1 \\ \hline 1 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \end{array}.$$

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\hline
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\hline
1 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1
\end{array}.$$

The following superposition is not valid, since it violates the condition **4:**

$$\begin{array}{ccccccc}
0 & 1 & \hat{2} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & \hat{1} & 1 & \hat{1} & 1 & \hat{1} & 1 \\
\hline
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[3] *If T is a tree on n vertices, then $U(T) \geq 2d - n$.*

2. The degree 2 vertices upper bound:

Theorem

[3] *For a linear tree T, $U(T) \leq 2 + D_2$, where $D_2(T)$ denotes the number of degree 2 vertices in T.*

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II. A formula for $U(T)$ for 2-linear trees (With M. Ingwersen)

III. Some preliminary results on $U(T)$ for nonlinear trees

I. Incremental changes of $U(T)$ for linear trees

Given a tree, we can add a vertex and obtain a larger tree via either adding a pendent vertex, in which a new edge and a vertex pendent at an existing vertex are added, or **edge subdivision**, in which a new vertex of degree 2 is positioned along an existing edge. When a tree is linear and the resulting tree upon vertex addition is also linear, we can use the LSP to prove that the change of $U(T)$ is bounded by 1.

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I. Incremental changes of $U(T)$ for linear trees

Theorem

Let T be a linear tree, and let T' be a linear tree obtained by adding a vertex to T, through either adding a pendent vertex or edge subdivison. Then $|U(T') - U(T)| \leq 1$.

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Proof. Given a linear tree $T = L(T_1, s_1, \cdots, s_k, T_{k+1})$, we may add a vertex in the following 6 ways while maintaining the linearity.

1 Add a pendent vertex to the endpoint of an arm of some T_i;

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4 Subdivide an edge on a connecting path s_i for some $1 \leq i \leq k$;

5 Add a pendent vertex to a vertex on a connecting path s_i;

6 Add a pendent vertex to a non-pendent vertex on an arm of a peripheral g-star (i.e T_1 or T_{k+1})

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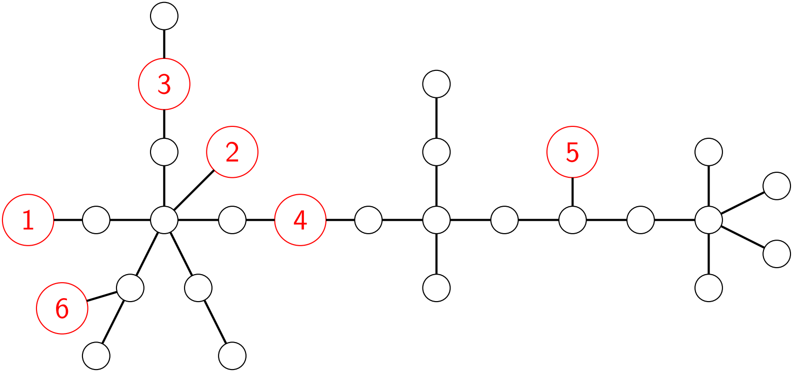

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Empirical evidence from the database [3]:

Table: Changes in $U(T)$ after adding a pendent vertex [4]

	-1	0	$+1$
Isolated	0	0	1
Pendent	221	430	554
Degree 2	936	85	0
HDV	226	594	0
Total	1383	1109	555

Table: Changes in $U(T)$ after edge subdivision

-1	0	$+1$
225	490	909

When a vertex is added in a particular way

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Table: Possible changes of $U(T)$ upon addition of a vertex

Vertex addition		+1	+0	-1
At an HDV		No	Yes	Yes
At a degree 2 vertex		No	Yes	Yes
At a pendent vertex	d increases by 1	Yes	Yes	Yes
	d stays the same	No	Yes	Yes
Edge subdivision		Yes	Yes	Yes

A new bound and a refined bound

A new diameter upper bound:

Corollary

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A refined $2 + D_2$ upper bound:

Theorem

For a linear tree $T = L(T_1, s_1, \cdots, s_k, T_{k+1})$, $U(T) = 2 + D_2$ if and only if T is **depth 1** *and either of the following is true:*

(a) $s_i = 0$ *for all* $1 \leq i \leq k$; *or*

(b) $deg_{T_j} = 3$ *for all* $1 < j < k + 1$.

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(b) $deg_{T_j} = 3$ for all $1 < j < k + 1$.

Corollary

If a linear tree T is not depth 1, then $U(T) \leq D_2 + 1$.

II. A formula for $U(T)$ for 2-linear trees

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$U(T)$ has been characterized for 1-linear trees, i.e. g-stars.

Prior result

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$$U(T) = \max\{1 + l_1, 2d(T) - n\}.$$

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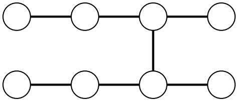

II. A formula for $U(T)$ for 2-linear trees

$U(T)$ has been characterized for 1-linear trees, i.e. g-stars.

Prior result

[1] Let T be a g-star with arm lengths $l_1 \geq \cdots \geq l_a$. Then

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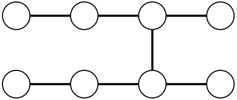

Now, moving one step further, we consider 2-linear trees in general.

II. A formula for $U(T)$ for 2-linear trees

Theorem

Let $T = L(T_1, s, T_2)$ be a 2-linear tree. Let n_i be the number of vertices in T_i, $i = 1, 2$. Let $l_1 \geq l_2 \geq \cdots \geq l_a$ (resp., $m_1 \geq m_2 \geq \cdots \geq m_b$) be the length of the arms of T_1 (resp., T_2). We also define $z(T_1) = \max\{0, l_1 - \sum_{i=2}^{i=a} l_i\}$ (resp., $z(T_2) = \max\{0, m_1 - \sum_{j=2}^{j=b} m_j\}$.) Then

$$U(T) = \max \begin{cases} 2 + z(T_1) + z(T_2) + s & \text{(upward } \hat{0} \text{ bound)} \\ l_1 + 1 - \lfloor \frac{n_2 - z(T_2) - 1}{2} \rfloor & \text{(difference bound for } T_1\text{)} \\ m_1 + 1 - \lfloor \frac{n_1 - z(T_1) - 1}{2} \rfloor & \text{(difference bound for } T_2\text{)} \\ 2d(T) - n_1 - n_2 - s & \text{(diameter bound)} \end{cases}$$

II. A formula for $U(T)$ for 2-linear trees

Example

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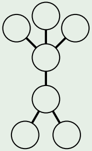

Upward $\hat{0}$ bound: **2**

Difference bounds: 1

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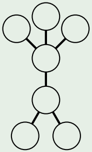

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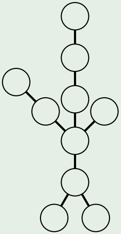

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II. A formula for $U(T)$ for 2-linear trees

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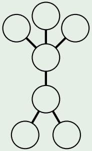

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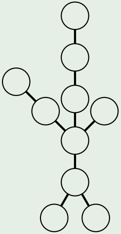

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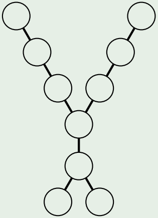

Upward $\hat{0}$ bound: 2

Difference bounds: 3

Diameter bound: **4**

$U(T) = 4$

III. Nonlinear trees

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The characterization of cores is that, a nonlinear tree T is a core for diameter d nonlinear trees if and only if $n(T) = d + 5$.

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4. The families of two different cores for some d might overlap with one another.
5. Cores of any diameter d nonlinear trees only contain 4 HDV's.

III. Nonlinear trees

Conjecture

Let T be a core of diameter d nonlinear trees, then

$$U(T) = \begin{cases} 2 & \text{if } d \leq 7 \\ d - 5 & \text{if } d \geq 8 \end{cases}.$$

Moreover, for any T' in the family generated by T, $U(T') \leq U(T)$.

III. Nonlinear trees

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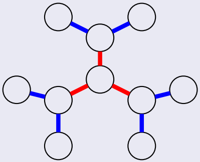
Algorithm

Start with the 10-vertex nonlinear tree, say T. There are two types of edges in T: R(ed) edges that are adjacent to the central vertex and B(lue) edges that are adjacent to some pendent vertex.



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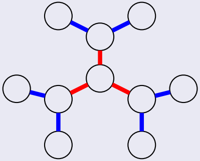

Since the cores are minimal nonlinear trees of diameter d, they are obtained by adding $d - 5$ vertices to T in a way such that the addition of each vertex increases the diameter by 1.

III. Nonlinear trees

Algorithm (Cont.)

In fact, up to isomorphism, it equates to doing edge subdivision $d-5$ times on one diameter of T. So, generating cores boils down to what we can do on one diameter of T.

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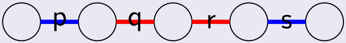

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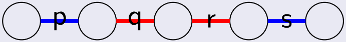

The set of all the cores of diameter d equals all possible nonnegative integer string partitions of $d-5$, i.e. $p+q+r+s=d-5$, up to isomorphism.

How many cores are there given a diameter d?

III. Nonlinear trees

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Proposition

Define $c(d)$ to be the number of cores for diameter d nonlinear trees. Then,

$$c(d) = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 6 & 12 \end{bmatrix}^T$$

where $p_i, 1 \leq i \leq 5$, denotes the number of non-isomorphic partitions of one of the following patterns: $(a, a, a, a), (a, b, b, b), (a, a, b, b), (a, b, b, c)$, and (a, b, c, d).

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In fact, the generating function of $c(d)$ is the expansion of

$$c(x+5) = \frac{(1+x^2)}{(1-x)^2(1-x^2)^2}.$$

III. Nonlinear trees

d	$c(d)$
5	1
6	2
7	6
8	10
9	19
10	28
11	44
12	60
13	85
14	110
15	146

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[2] C.R. Johnson, A. Li, and A. Walker, *Ordered multiplicity lists for eigenvalues of symmetric matrices whose graph is a linear tree*, Discrete Mathematics **333** (2014), 39–55.

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