Topics on the <u>N</u>onnegative <u>I</u>nverse <u>E</u>igenvalue <u>P</u>roblem

Charles Johnson

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The Nonnegative Inverse Eigenvalue Problem (NIEP)

For each n, which collections of n complex numbers (repeats allowed) occur as the eigenvalues of some n-by-n nonnegative matrix?

$(\underset{n\times n}{A} \geq 0)$ $\{\lambda_1, \lambda_2, \cdots \lambda_n\} = \sigma(A)$

Several necessary conditions follow from the P-F theory, eg

- $\rho(A) \in \sigma(A)$: $\lambda_1 \geq 0$; $\lambda_1 \geq |\lambda_i|$, $i=2,\cdots,n$
- $\lambda_i \in \sigma(A) \implies \overline{\lambda_i} \in \sigma(A)$ (mults.)
- $\sum \lambda_i = \mathrm{Tr}(A) \geq 0$
- $\sum \lambda_i^k = \mathrm{Tr}(A^k) \geq 0$

etc (eg JLL)

Not sufficient; realizable spectra ?

Recent survey: Marijuan, Paparella, Pisonero, CT

Aside

If Λ meets the necessary conditions to be the spectrum of a primitive matrix, Λ may not be realizable, but suffiently many 0's may be appended to Λ, so that $\Lambda \cup \{0, 0, \ldots, 0\}$ is realizable! (Boyle & Handelman, symbolic dynamics)

Example. $1, \pm\sqrt{\frac{3}{8}}\, i$ is not realizable for $n=3$, but $1, \pm\sqrt{\frac{3}{8}}\, i, 0$ is for $n=4$

$$A = \frac{1}{24}\begin{bmatrix} 6 & 0 & 18 & 0 \\ 6 & 6 & 0 & 12 \\ 0 & 18 & 6 & 0 \\ 7 & 0 & 11 & 6 \end{bmatrix}$$

CJ 1976

$$\underset{n\times n}{B} = \underset{n\times m}{C}\,\underset{m\times n}{D} \longrightarrow \underset{m\times m}{B'} = DC, \quad m \geq n$$

Many (interesting) variations, sub-problems

Require
- eigenvalues to be real
- A to be symmetric
- A to be diagonalizable
- A to be doubly stochastic
- A to be irreducible
- A to be primitive

Allow A to have negative entries, but require it to have eventually positive powers

Which Jordan structures can occur

etc.

① DS single eigenvalue problem

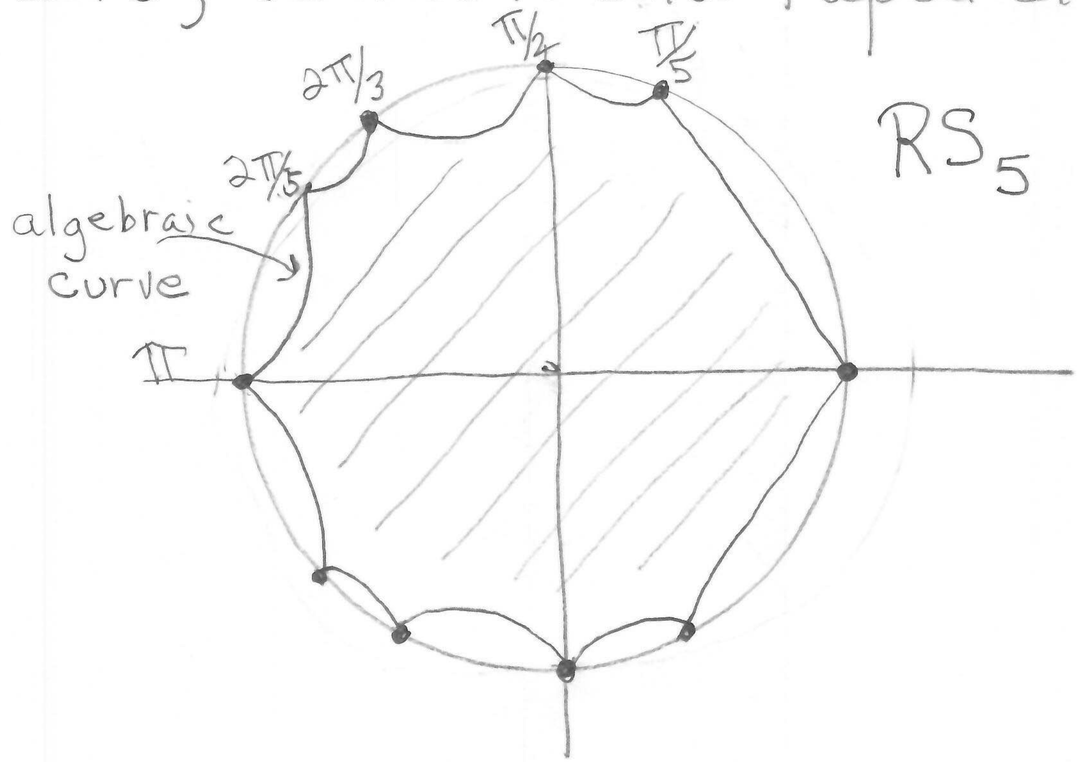
② Possible Jordan structure in realizations.

- The Doubly Stochastic Single Eigenvalue Problem

Which individual complex numbers occur as an eigenvalue of a DS matrix? $|\lambda| \leq 1 = \rho(A)$

Recall: Row Stochastic case

Dmitriev and Dynkin, Karpelevich, Ito, Johnson and Paparella

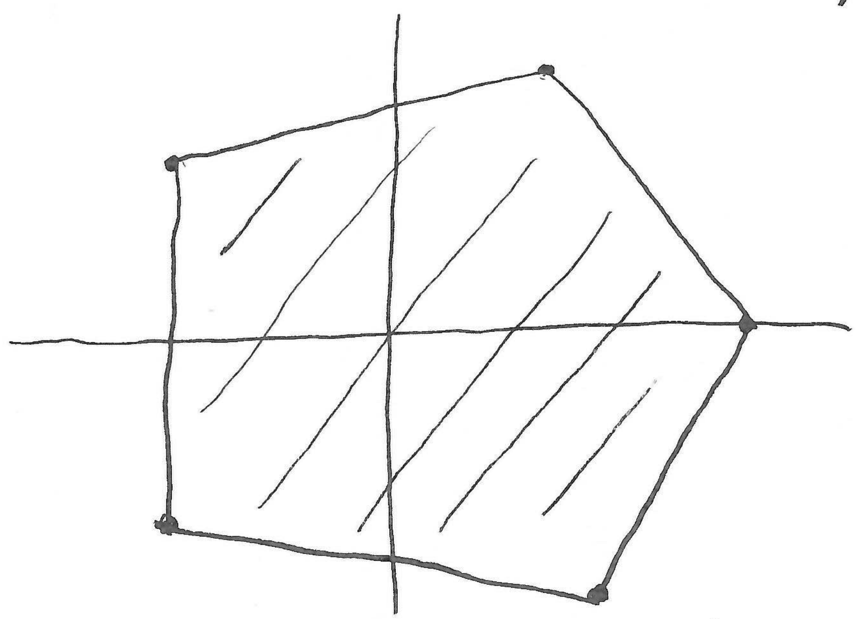


What about DS_n?

H Perfect, L Mirsky 1965

$P_k = \mathrm{Co}\{k\text{-th roots of unity}\}$

P_5



Let $PM_n = \bigcup_{k=2}^{n} P_k$ jagged boundary piecewise linear

<u>Theorem</u> $PM_n \subseteq DS_n$

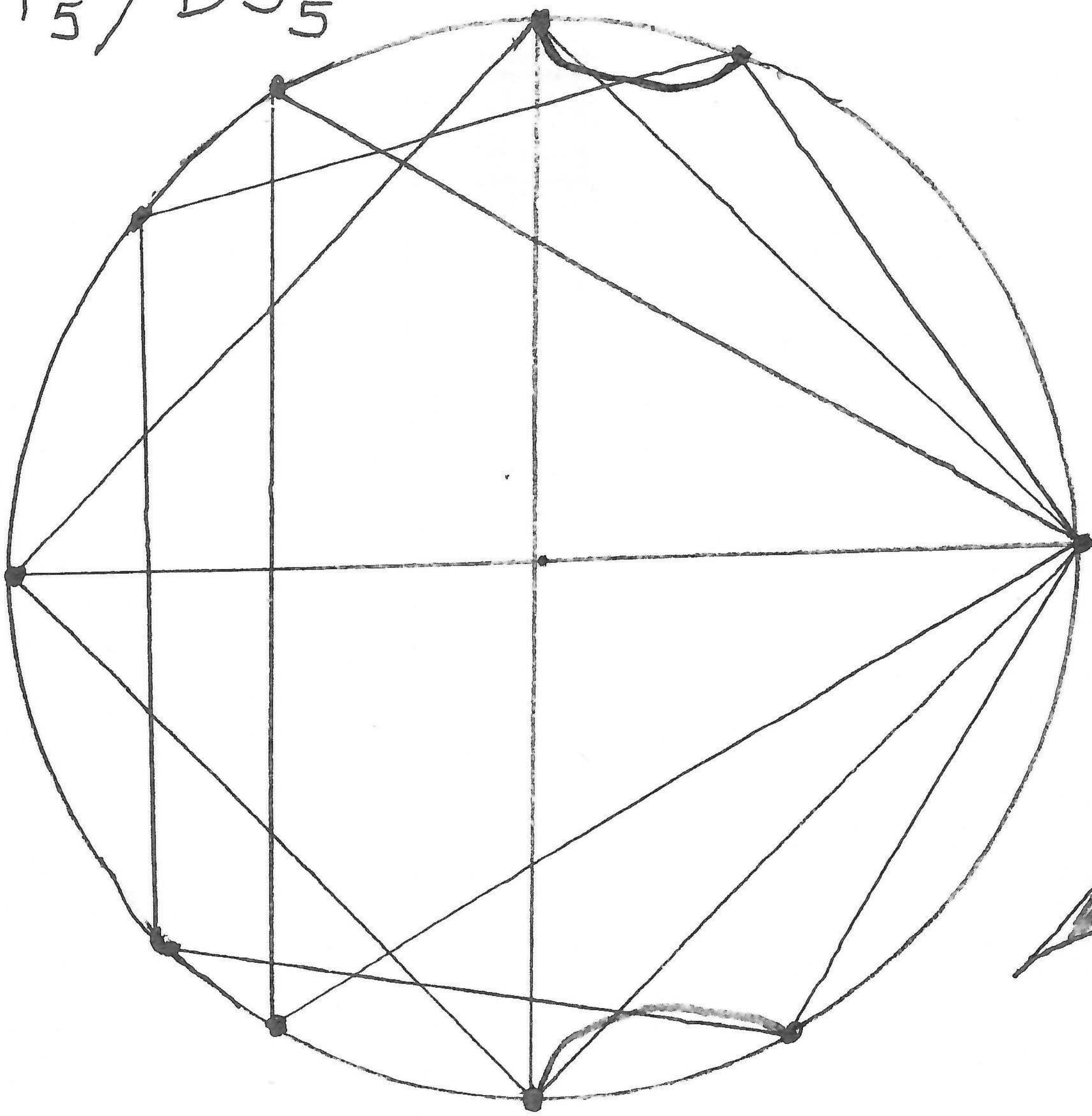
Are they equal? "P-M conjecture"

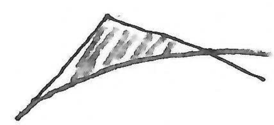
Yes $n \le 3$ (P-M)

Yes $n = 4$ (Levick, Pereira, Kribs, 2014)

No $n = 5$ (Mashreqi, Rivard, 2007)

PM_5 / DS_5





$$t \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} + (1-t) \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

DS for $t \in [0, 1]$

an eigenvalue passes outside PM_5 for $t \in [.47, 55]$ apprx.

Several other pairs of permutations produce the same curve, due to permutation similarity.

But, to a very high degree of precision, there is no other 5-by-5 DS matrix producing eigenvalues in DS_5, outside PM_5 anywhere else.
(A. Harlev, CJ, D Lim, to appear)

- DS_n is "solid" star-shaped from 0, 1

  So, the boundary suffices
- Recall: the DS matrices are the convex hull of the permutation matrices (and $(n-1)^2+1$ suffice)
- So, finitely many collections of permutations generate all DS matrices

But $\binom{n!}{(n-1)^2+1}$ grows very fast!

Fortunately, many fewer suffice.

With a rather fine mesh, this allows approximate consideration of all DS matrices, for $n < 12$

Eigenvalues of each may be computed rapidly and accurately

Storage of boundary eigenvalues not difficult.

So, the PM conjecture may be checked, rather reliably.

What was found.

- no other counterexamples for $n=5$ (very likely) (refined mesh may be used near boundary)
- no counterexamples for $n<12$ (likely higher, via spot-checking)
- Big idea: Boundary <u>Conjecture</u>

  All boundary points of DS_n are eigenvalues of DS matrices that are convex combinations of at most 2 permutations

  !!

This was checked computationally for $n < 12$ and seems to be correct. Should be provable. Ideas?

If the Boundary Conjecture is correct, P-M can be checked further ($\sim n = 20$) eg $n = 10$, $< 2M$ "inequivalent" pairs of permutations need be checked (group theory helps a lot)

Which pairs determine the boundary?

Note: for n (not very) large, DS_n is apprx the unit disc (limit as $n \to \infty$), as $PM_n \longrightarrow$ unit disc, $n \to \infty$.

So, one might expect that counterexamples to

$$DS_n = PM_n$$

are more likely for smaller n.

Related general problems

G a matrix group

$H = Co(G)$, a semigroup.

If G is finite, similar methods may be used to investigate H.

Boundary conjecture ?

Possible Jordan Structure

$\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is <u>realizable</u>
if there is an $\underset{n \times n}{A} \geq 0$
with $\sigma(A) = \Lambda$ (repeats allowed)

<u>Diagonalizably</u> <u>realizable</u> if
there is a diagonalizable A.

Of course, if the λ_i are distinct,
realizable = diag. realizable (DR)

- Assume there are repeats (not including the Perron root) (recent A. Julio, R. Soto, CJ)
- Λ is <u>universally</u> <u>realizable</u> (UR) if all Jordan forms allowed by Λ occur among its realizations

<u>Remark</u>: It can happen that Λ is realizable, but not DR!

$\{a, -\frac{a}{4} \pm \frac{\sqrt{5}a}{4}i, -\frac{a}{4} \pm \frac{\sqrt{5}a}{4}i\}$

So, what if Λ is DR, must other JCF's be possible?

Some positive results.

H. Minc observation

- If Λ is realizable by a <u>positive</u> matrix, then Λ is UR.

Easy: Any sufficiently small perturbation in the JCF will keep $A > 0$

$$0 < A = S^{-1} \underbrace{\begin{bmatrix} \bullet & & & & & & \\ & \bullet & & & & & \\ & & \bullet & & & & \\ & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & \bullet & \\ & & & & & & \lambda & \square \\ & & & & & & & \lambda \end{bmatrix}}_{J} S$$

$$A = S^{-1}(J + \varepsilon E_{n-1,n})S$$

$$= A + \varepsilon S^{-1} E_{n-1,n} S > 0$$

for $\varepsilon > 0$ small

but the JCF, not the eigenvalues changes with small perturbation

Several Particular Results

eg Suleimanova spectra

$1;\ \lambda_2, \lambda_3, \cdots, \lambda_n \leq 0$

$1 + \sum_{2}^{n} \lambda_i \geq 0$

realizable (S, 1949), <u>also UR</u>

ODP realizable
(all off-diag entries > 0)

or irreducible with an
off-diagonal line > 0

DR $\Longrightarrow$ UR.

Does DR $\Longrightarrow$ UR always?

Permutative $\Longrightarrow$

Suleimanova

$1, -\lambda_2, -\lambda_3, -\lambda_4 \qquad \lambda_i \geq 0$

$i = 2, 3, 4$

$\sum \lambda_i = 1$

$$\det\left(\begin{bmatrix} 0 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2 & 0 & \lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_3 & 0 & \lambda_4 \\ \lambda_2 & \lambda_3 & \lambda_4 & 0 \end{bmatrix} + xI\right) = 0$$

$x = -1, \lambda_2, \lambda_3, \lambda_4$

So, eigenvalues

$1, -\lambda_2, -\lambda_3, -\lambda_4$

Threshhold Facts

Suppose $\Lambda = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$ is realizable with $\lambda_1 = \max_{1 \le i \le n} |\lambda_i|$.

Known: $\Lambda_t = \{\lambda_1 + t, \lambda_2, \cdots, \lambda_n\}$, $t > 0$, is also realizable.

So, there is a minimum value $g = g(\lambda_2, \lambda_3, \cdots, \lambda_n)$ such that $\Lambda_g = \{g, \lambda_2, \cdots, \lambda_n\}$ is realizable and $\Lambda_{g'}$ is realizable for all $g' \ge g$.

<u>Proof</u>: Brauer's rank 1 perturbation thm

<u>Note</u>: Understanding $g(\lambda_2, \cdots, \lambda_n)$ is the NIEP.

There are similar results for the D-NIEP or the NIEP associated with any (fixed) Jordan structure among $\lambda_2, \lambda_3, \cdots, \lambda_n$

diagonalizable

Let g_d be the diagonalizable threshhold value

Then $g_d \geq g$ and equality is possible (even when the λ_i's are not distinct)

<u>Open question</u>: For which $\lambda_2, \lambda_3, \cdots, \lambda_n$ is $g_d = g$?

<u>Theorem</u> If $\Lambda = \{g_d, \lambda_2, \cdots, \lambda_n\}$ and $\lambda_1 > g_d$, then

$$\Lambda' = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$$

is UR.

Almost proves DR $\Longrightarrow$ UR

No counterexamples known.

Further conjecture:

If Λ is realizable with a given Jordan structure, Λ is also realizable with any coarser Jordan structure

(similar partial results.)