COHERENT CONFIGURATIONS & QUANTUM ORBITAL ALGEBRAS

(Based on work of Lupini, Mančinska, Roberson (2020))

DEF Given a set Ω, a <u>coherent configuration</u> is a partition $\mathcal{R} = \{R_i : i \in I\}$ of $\Omega^2 = \Omega \times \Omega$ into <u>relations</u> (or <u>classes</u>) satisfying:

(1) There is a subset $D \subseteq I$ of the index set such that $\{R_d : d \in D\}$ is a partition of the <u>diagonal</u> $\{(\alpha,\alpha) : \alpha \in \Omega\}$. These are called the <u>diagonal relations</u>.

(2) For each relation R_i, its <u>converse</u> $\{(\beta,\alpha) : (\alpha,\beta) \in R_i\}$ is also a relation, say $R_i^{-1} \in \mathcal{R}$. ($R_i^{-1} = R_i$ is allowed!)

(3) For all $i,j,k \in I$ and $\forall\, (\alpha,\beta) \in R_k$, the number of $\gamma \in \Omega$ such that $(\alpha,\gamma) \in R_i$ and $(\gamma,\beta) \in R_j$ is a <u>constant</u> p_{ij}^k that does <u>not</u> depend on α & β.

p_{ij}^k: <u>intersection numbers</u> of $\mathcal{R}$

Why would make such a definition?

Donald Higman defined coherent configurations in an attempt to "do group theory without groups"!

<u>EX</u> Let $G \leq \mathrm{Sym}(\Omega)$.

Then, G has a natural action on pairs:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g) \quad , \quad g \in G$$

<u>NOTATION</u> $\alpha^g = \gamma$ means g sends α to γ!

Define a relation $\sim_G$ on $\Omega \times \Omega$ by

$(\alpha, \beta) \sim_G (\gamma, \delta)$ if $\exists g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$.

<u>CLAIM</u> $\sim_G$ is an equivalence relation on $\Omega \times \Omega$

<u>REFLEXIVE</u> If $1 \in G$ is the identity, then $(\alpha, \beta)^1 = (\alpha, \beta)$ ✓

SYMMETRIC Suppose $(\alpha, \beta) \sim_G (\gamma, \delta)$.

Then $(\alpha, \beta)^g = (\gamma, \delta)$

$$\Rightarrow (\gamma, \delta)^{g^{-1}} = \left((\alpha, \beta)^g\right)^{g^{-1}} = (\alpha, \beta)^{(gg^{-1})} = (\alpha, \beta) \checkmark$$

TRANSITIVE Suppose $(\alpha_1, \beta_1) \sim_G (\alpha_2, \beta_2)$, $(\alpha_2, \beta_2) \sim_G (\alpha_3, \beta_3)$.

Then, $\exists\, g, h \in G$ such that

$$(\alpha_1, \beta_1)^g = (\alpha_2, \beta_2), \quad (\alpha_2, \beta_2)^h = (\alpha_3, \beta_3)$$

So: $(\alpha_1, \beta_1)^{gh} = (\alpha_3, \beta_3)$ $\checkmark$

So: $\sim_G$ is an equivalence relation on $\Omega \times \Omega$

$\Rightarrow$ equivalence classes are a partition of $\Omega \times \Omega$

Let this partition be $\mathcal{R}_G = \{ R_i : i \in I \}$

orbit: $\alpha^G = \{\alpha^g : g \in G\}$

each R_i is an equivalence class of $\Omega \times \Omega$ under $\sim_G$

orbital: $(\alpha, \beta)^G = \{(\alpha, \beta)^g = (\alpha^g, \beta^g) : g \in G\}$

Called orbitals

CLAIM $\mathcal{R}_G$ is a coherent configuration.

① ("Partition of the diagonal")

For each $(\alpha, \alpha) \in \Omega^2$, $g \in G$,

$(\alpha, \alpha)^g = (\alpha^g, \alpha^g)$ is also in the diagonal, i.e.,

if $(\alpha, \alpha) \in R_d$ and $\mathcal{D} := \{(\gamma, \gamma) : \gamma \in \Omega\}$,

then $R_d \subseteq \mathcal{D}$.

In fact, if α^G is the orbit of α under G and

$(\alpha, \beta)^G := \{(\alpha, \beta)^g : g \in G\}$,

then $(\alpha, \alpha)^G = \{(\beta, \beta) : \beta \in \alpha^G\}$.

this is a whole relation/class of $\mathcal{R}$!

(if $(\alpha, \alpha) \in R_d$, then $R_d = (\alpha, \alpha)^G$)

② (Converse relation)

Let $R_i = \{(\alpha, \beta)^g : g \in G\} = (\alpha, \beta)^G$

Define $R_i' := \{(\beta, \alpha)^g : g \in G\} = (\beta, \alpha)^G$.

Then, $(\gamma, \delta) \in R_i \Rightarrow (\gamma, \delta) = (\alpha, \beta)^g$ for some $g \in G$

$\Rightarrow \gamma = \alpha^g, \quad \delta = \beta^g$

$\Rightarrow \quad (\delta, \gamma) = (\beta, \alpha)^g \in R_{i'}$

So, $(\beta, \alpha)^g$ is the converse of $(\alpha, \beta)^g$.

(3) ("intersection numbers")

Let $(\alpha, \beta) \in R_k$, and assume there are exactly p_{ij}^k elements $\gamma \in \Omega$ such that $(\alpha, \gamma) \in R_i$, $(\gamma, \beta) \in R_j$.

Then, for any $(\alpha', \beta') \in R_k$, $(\alpha', \beta') = (\alpha, \beta)^g$

First, note that if $(\alpha, \gamma) \in R_i$, $(\gamma, \beta) \in R_j$,

then $(\alpha^g, \gamma^g) = (\alpha, \gamma)^g \in R_i$,

$(\gamma^g, \beta^g) = (\gamma, \beta)^g \in R_j$,

So there are $\geq p_{ij}^k$ such γ''s w/ $(\alpha', \gamma') \in R_i$, $(\gamma', \beta') \in R_j$.

Second, if $(\alpha', \gamma') \in R_i$, $(\gamma', \beta') \in R_j$,

then $(\alpha', \gamma')^{g^{-1}} = (\alpha, (\gamma')^{g^{-1}}) \in R_i$,

$(\gamma', \beta')^{g^{-1}} = ((\gamma')^{g^{-1}}, \beta) \in R_j$,

so there are $\leq p_{ij}^k$ such γ''s.

so: exactly p_{ij}^k such γ''s! ✓

EQUIVALENTLY: Given any coherent configuration $\mathcal{R} = \{R_i : i \in I\}$ on Ω, one can construct a matrix A_i for each $i \in I$:

$$(A_i)_{\alpha\beta} := \begin{cases} 1, & \text{if } (\alpha,\beta) \in R_i \\ 0, & \text{otherwise} \end{cases}$$

$\{A_i : i \in I\}$: <u>characteristic matrices</u> of the configuration

<u>**PROP**</u> The linear span of the $\{A_i : i \in I\}$, i.e., the set of all matrices of the form

$$\sum_{i \in I} c_i A_i \qquad (c_i \in \mathbb{C})$$

is a self-adjoint, unital algebra containing the all-ones matrix J and which is closed under $\circ$, the <u>entrywise</u> product.

<u>Pf</u> By construction, it's closed under addition & scalar mult.

Since $A_i \circ A_j = \delta_{ij} A_i = \begin{cases} 0, & \text{if } i \neq j, \\ A_i, & \text{if } i = j \end{cases}$

it's closed under $\circ$.

It's unital, since $I = \sum_{d \in D} A_d$,

and $J = \sum_{i \in I} A_i$ (partition!) $\square$

<u>DEF</u> This algebra $\mathcal{A}(\mathcal{R})$ is called a <u>coherent algebra</u>.

<u>PROP</u> For all $i, j \in I$,

$$A_i A_j = \sum_{k \in I} p_{ij}^k A_k .$$

<u>matrix</u> mult

Consequently, $\mathcal{A}(\mathcal{R})$ is closed under matrix multiplication.

<u>Pf</u> $(A_i A_j)_{\alpha\beta} = \sum_{\gamma \in \Omega} (A_i)_{\alpha\gamma} (A_j)_{\gamma\beta}$

So, if $(\alpha,\beta) \in R_k$, $(A_i A_j)_{\alpha\beta}$ is the number of γ such that $(\alpha,\gamma) \in R_i$ & $(\gamma,\beta) \in R_j$,

i.e., $(A_i A_j)_{\alpha\beta} = p_{ij}^k$. $\square$

<u>NOTE</u> Since any coherent algebra is closed under the entrywise product, there's a unique basis of 0,1-matrices which define a partition of $\Omega \times \Omega$

so: $\left\{ \begin{array}{c} \text{coherent} \\ \text{configurations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{coherent} \\ \text{algebras} \end{array} \right\}$

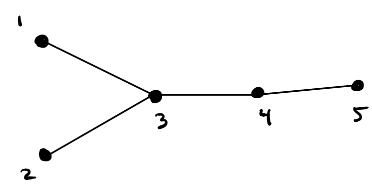
<u>DEF</u> Given a graph Γ w/ adjacency matrix A, we can define two (not necessarily distinct!) coherent algebras (on $V(\Gamma) \times V(\Gamma)$):

(1) $\mathcal{Q}(\mathcal{R}_\Gamma)$, the <u>coherent algebra</u> of Γ, which is the unique smallest coherent algebra containing A (and I and J).

(2) $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$, the <u>orbital algebra</u> of Γ, which is the coherent algebra corresponding to the relation $\sim_{Aut(\Gamma)}$ on $V(\Gamma) \times V(\Gamma)$

<u>NOTE</u> $A \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$, so $\mathcal{A}(\mathcal{R}_\Gamma) \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$

<u>Ex 1</u> Γ:



$$Aut(\Gamma) = \langle (1\,2) \rangle \quad (\cong S_2 \cong C_2)$$

<u>orbitals</u>: $\underbrace{\{(1,1),(2,2)\}}_{R_1}$, $\{(3,3)\}$, $\{(4,4)\}$, $\{(5,5)\}$,

$\{(1,2),(2,1)\}$, $\{(1,3),(2,3)\}$, $\underbrace{\{(3,4)\}}_{R_6}$

etc.

Exactly 17 orbitals

$$A_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

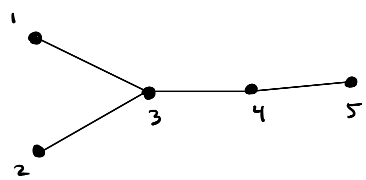
etc.

So: $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$ generated by these 17 matrices

(17-dim v.s.)

On the other hand, if

Γ:



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

then, actually, the smallest coherent algebra containing A, I, J is $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$!

so: $$\mathcal{A}(\mathcal{R}_\Gamma) = \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$$

<u>NOTE</u> There is an efficient (polynomial time) algorithm for computing $\mathcal{A}(\mathcal{R}_\Gamma)$:

Weisfeiler - Leman Algorithm

(Lehman)

each vertex has 12 neighbors

every pair of nonneighbors has 6 common neighbors

<u>EX 2</u> There exists a $(25, 12, 5, 6)$ - strongly regular graph

vertices

every pair of neighbors have 5 common neighbors

$\mathcal{C}$ (a Chang graph) w/ $Aut(\mathcal{C}) = \{1\}$.

So, $\mathcal{A}(\mathcal{R}_{Aut(\mathcal{C})}) = M_{25}(\mathbb{C})$

(for every $\alpha, \beta \in V(\mathcal{C})$, $\{(\alpha,\beta)\}$ is its own orbital).

$$\mathrm{Dim}_{\mathbb{C}}\left(\mathcal{A}(\mathcal{R}_{Aut(\mathcal{C})})\right) = 25^2 = 625$$

On the other hand, if A is the adjacency matrix of $\mathcal{C}$,

$$A^2 = 12I + 5A + 6(J - I - A),$$

so $\{A, I, J\}$ is a basis for $\mathcal{A}(\mathcal{R}_\Gamma)$

$$\mathrm{Dim}_{\mathbb{C}}\left(\mathcal{A}(\mathcal{R}_\Gamma)\right) = 3$$

These coherent algebras are very different!

RECALL:

Automorphisms of Γ $\rightsquigarrow$ Commutative C^*-algebra $\xrightarrow{\text{Liberation}}$ Quantum Automorphisms

$Aut(\Gamma)$ $\quad$ $C(Aut(\Gamma))$ $\quad$ $C(Aut(\Gamma)^+)$, $Aut(\Gamma)^+$

magic unitary: $U = (u_{ij})$ element in some unital C^*-algebra $(n \times n)$

(1) Each u_{ij} is a projection $u_{ij}^2 = u_{ij}^* = u_{ij}$

(2) "Partition of unity": $\sum_{l=1}^{n} u_{il} = \sum_{l=1}^{n} u_{li} = 1$,

$$u_{ij}u_{ik} = \delta_{jk}u_{ij}, \quad u_{ji}u_{ki} = \delta_{jk}u_{ji}$$

((3) U is unitary: $UU^* = I = U^*U$)

Quantum permutation group:

$$S_n^+ \longleftrightarrow C(S_n^+) := \langle u_{ij} : \text{satisfy (1) \& (2)} \rangle$$

If Γ has $n \times n$ adjacency matrix A, then we define the quantum automorphism group $Aut(\Gamma^+)$ (or $Qut(\Gamma)$) by

$$Aut(\Gamma)^+ \longleftrightarrow C(Aut(\Gamma)^+) := C(S_n^+) / \langle AU = UA \rangle$$

Since $C(Aut(\Gamma)) \cong C(Aut(\Gamma)^+) / \langle MN = NM \rangle$,

$$Aut(\Gamma) \leq Aut(\Gamma)^+.$$

Relations: If α & γ are adjacent in Γ
and β & δ are not adjacent in Γ,

Then: $u_{\alpha\beta} u_{\gamma\delta} = u_{\gamma\delta} u_{\alpha\beta} = u_{\beta\alpha} u_{\delta\gamma} = u_{\delta\gamma} u_{\beta\alpha} = 0$.

(IDEA: can't map edges to nonedges!)

NEW IDEA: (Lupini, Mančinska, Roberson (2020))

We can define an equivalence relation $\sim_2$ on $V(\Gamma) \times V(\Gamma)$ by:

$(\alpha, \beta) \sim_2 (\gamma, \delta)$ if $u_{\alpha\gamma} u_{\beta\delta} \neq 0$

(Note also that $\sim_1$ on $V(\Gamma)$ defined by
$\alpha \sim_1 \beta$ if $u_{\alpha\beta} \neq 0$ also equivalence relation)

Equivalence classes of $\sim_2$: quantum orbitals

THM (LMR 2020) The quantum orbitals of Γ form a coherent configuration $\mathcal{R}_{Aut(\Gamma)^+}$ w/ corresponding coherent algebra $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)^+})$.

NOTE • If $\alpha, \beta \in V(\Gamma)$ are in the same classical orbital, they are in the same quantum orbital. So,

$$\mathcal{A}(\mathcal{R}_{Aut(\Gamma)^+}) \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$$

• $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)^+})$ must contain the adjacency matrix A of Γ!

($(\alpha, \beta) \sim_2 (\gamma, \delta)$ is not possible if one pair is an edge, other isn't!)

So,

$$\mathcal{A}(\mathcal{R}_\Gamma) \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)^+}) \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$$

THM (LMR 2020) Let Γ be a graph w/ adjacency matrix A. If U is a magic unitary that commutes w/ A, then:

(1) U commutes w/ every element of $\mathcal{A}(\mathcal{R}_\Gamma)$;

(2) $u_{\alpha\beta} u_{\gamma\delta} = 0$ unless $(\alpha, \gamma), (\beta, \delta)$ are both in the same class of $\mathcal{R}_\Gamma$.

THM (LMR 2020) Almost all graphs on n vertices have trivial quantum automorphism group as $n \longrightarrow \infty$.

Pf • Babai, Kucera (1979): almost all graphs have trivial coherent configuration $\mathcal{R}_\Gamma$, i.e.,

$$\mathcal{R}_\Gamma = \{ \{(\alpha,\beta)\} : \alpha, \beta \in V(\Gamma) \}$$

• Thus, $M_n(\mathbb{C}) = \mathcal{Q}(\mathcal{R}_\Gamma) \subseteq \mathcal{Q}(\mathcal{R}_{Aut(\Gamma)^+}) \subseteq \mathcal{Q}(\mathcal{R}_{Aut(\Gamma)}) \subseteq M_n(\mathbb{C})$,

so $\mathcal{Q}(\mathcal{R}_{Aut(\Gamma)^+}) = M_n(\mathbb{C})$.

and $Aut(\Gamma) = \{1\}$.

• So, unless $\alpha = \gamma$ and $\beta = \delta$,

$$u_{\alpha\beta} u_{\gamma\delta} = 0 ;$$

otherwise, $u_{\alpha\beta} u_{\alpha\beta} = u_{\alpha\beta}$.

• Therefore, $C(Aut(\Gamma)^+)$ is commutative, and

so $Aut(\Gamma)^+ = Aut(\Gamma) = \{1\}$. $\square$