INTRODUCTION TO QUANTUM GROUPS IV:

COHERENT CONFIGURATIONS & QUANTUM ORBITAL ALGEBRAS

(Based on work of Lupini, Mančinska, Roberson (2020))

DEF Given a set Ω, a coherent configuration is a partition $\mathcal{R} = \{R_i : i \in I\}$ of $\Omega^2 := \Omega \times \Omega$ into relations (or classes) satisfying:

(1) ("Diagonal") There is a subset $D \subseteq I$ of the index set such that $\{R_d : d \in D\}$ is a partition of the diagonal $\{(\alpha,\alpha) : \alpha \in \Omega\}$. (These are called the diagonal relations.)

(2) ("Converse") For each relation R_i, its converse $\{(\beta,\alpha) : (\alpha,\beta) \in R_i\}$ is also a relation, say $R_{i'} \in \mathcal{R}$.

($R_{i'} = R_i$ is allowed!)

(3) ("Intersection Numbers")

For all $i, j, k \in I$ and any $(\alpha, \beta) \in R_k$,

the number of $\gamma \in \Omega$ such that

$(\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j$ is a

constant p_{ij}^k that does not depend

on α & β.

p_{ij}^k : intersection numbers of $\mathcal{R}$

Q : Why make such a definition?

A : Donald Higman defined coherent

configurations (CC's) in an attempt

to "do group theory without groups"!

EX Let $G \leq \mathrm{Sym}(\Omega)$

NOTATION: If $\alpha, \beta \in \Omega$ and g sends α to β, then we write

$$\alpha^g = \beta$$

orbit of α under G:

$$\alpha^G := \{ \alpha^g : g \in G \}$$

Natural action on pairs in $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g)$$

The orbit $(\alpha, \beta)^G$ under this action on pairs is called an orbital.

Define a relation $\sim_G$ on $\Omega \times \Omega$ by

$(\alpha, \beta) \sim_G (\gamma, \delta)$ if $\exists g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$

EXERCISE: $\sim_G$ is an equivalence relation on $\Omega^2 = \Omega \times \Omega$

so: the equivalence classes of $\Omega \times \Omega$ under $\sim_G$ partition Ω^2
(orbitals)

Let this partition be

$$\mathcal{R}_G := \{ R_i : i \in I \}$$

an orbital $(\alpha, \beta)^G$

CLAIM $\mathcal{R}_G$ is a coherent configuration

(1) Partition of diagonal?

Yes. $(\alpha, \alpha)^G \longleftrightarrow \{ (\beta, \beta) : \beta \in \alpha^G \}$ ✓

(2) Converse? Suppose $R_i = (\alpha, \beta)^G$

Let $(\gamma, \delta) \in (\alpha, \beta)^G$

$\Rightarrow (\gamma, \delta) = (\alpha, \beta)^g$

so, $\alpha^g = \gamma, \quad \beta^g = \delta$

$\Rightarrow (\delta, \gamma) \in (\beta, \alpha)^g$ ✓

③ Intersection numbers?

Start w/ $(\alpha, \beta) \in R_k$

$$\alpha - \underbrace{\gamma \overbrace{- \beta}^{(\gamma,\beta) \in R_j}}_{(\alpha,\gamma) \in R_i}$$

Let $(\alpha', \beta') \in R_k \implies \alpha' = \alpha^g, \ \beta' = \beta^g$

for some $g \in G$

$$\alpha^g - \underbrace{\gamma^g \overbrace{- \beta^g}^{(\gamma^g, \beta^g) \in R_j}}_{(\alpha^g, \gamma^g) \in R_i}$$

Conversely, $\alpha' - \gamma' - \beta'$

$$\Downarrow$$

$$\alpha - (\gamma')^{g^{-1}} - \beta \quad \checkmark$$

The number p_{ij}^k of γ''s is

the same for (α', β') as

the # of γ's for (α, β) ✓

EQUIVALENTLY: Given any CC

$\mathcal{R} = \{ R_i : i \in I \}$ on Ω, one

can construct a matrix A_i for

each $i \in I$:

$$(A_i)_{\alpha\beta} := \begin{cases} 1, & \text{if } (\alpha,\beta) \in R_i \\ 0, & \text{otherwise} \end{cases}$$

$\{ A_i : i \in I \}$: <u>characteristic matrices</u> of the CC

PROP The linear span of the $\{ A_i : i \in I \}$,

i.e., the set of all matrices

$$\sum_{\substack{i \in I \\ (\text{finite})}} c_i A_i \quad (c_i \in \mathbb{C})$$

is a self-adjoint, unital algebra

containing the all-ones matrix J

and which is closed under $\circ$;

The entrywise product.

DEF This algebra $\mathcal{A}(\mathcal{R})$ is called a coherent algebra.

PROP For all $i, j \in I$,

$$\underbrace{A_i A_j}_{\text{matrix mult.}} = \sum_{k \in I} p_{ij}^k A_k$$

Consequently, $\mathcal{A}(\mathcal{R})$ is closed under matrix multiplication.

"PF" $$(A_i A_j)_{\alpha\beta} = \sum_{\gamma \in \Omega} (A_i)_{\alpha\gamma} (A_j)_{\gamma\beta}$$

NOTE Since any coherent algebra is closed under the entrywise product, there's a unique basis of 0,1-matrices which define a partition of $\Omega \times \Omega$

so: $\left\{ \begin{array}{c} \text{coherent} \\ \text{configurations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{coherent} \\ \text{algebras} \end{array} \right\}$

<u>DEF</u> Given a graph Γ w/ adjacency matrix A, we can define two (not necessarily distinct!) coherent algebras on $V(\Gamma) \times V(\Gamma)$:

(1) $\mathcal{A}(\mathcal{R}_\Gamma)$, the <u>coherent algebra</u> of Γ, which is the unique smallest coherent algebra containing A (and I and J).

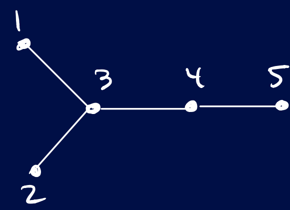
(2) $\mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$, the <u>orbital algebra</u> of Γ, which is the coherent algebra corresponding to the relation $\sim_{Aut(\Gamma)}$ on $V(\Gamma) \times V(\Gamma)$.

<u>NOTE</u> $A \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$, so $\mathcal{A}(\mathcal{R}_\Gamma) \subseteq \mathcal{A}(\mathcal{R}_{Aut(\Gamma)})$

NOTE There is an efficient (polynomial time) algorithm for computing $\mathcal{A}(\mathcal{R}_\Gamma)$:

Weisfeiler – Leman (Lehman) Algorithm

Ex 1 Γ:



$\mathrm{Aut}(\Gamma) = \langle (1,2) \rangle$

Orbitals: (17 of them)

$\{(1,1),(2,2)\}, \ \{(3,3)\}, \ \ldots$

$\{(1,3),(2,3)\}, \ \{(3,4)\}, \ \ldots$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathcal{A}(\mathcal{R}_\Gamma)$: Dimension 17 over $\mathbb{C}$!

As it turns out, $\mathcal{A}(\mathcal{R}_\Gamma) = \mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\Gamma)})$

EX 2 There is $(25, 12, 5, 6)$-strongly regular graph $\mathcal{C}$ (a Chang graph) 25 vertices

w/ $\mathrm{Aut}(\mathcal{C}) = \{1\}$

$$\mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\mathcal{C})}) = M_{25}(\mathbb{C})$$

Dimension 625 over $\mathbb{C}$

(for every $\alpha, \beta \in V(\mathcal{C})$, $\{(\alpha, \beta)\}$ is its own orbital)

On the other hand, if A is the adjacency matrix of $\mathcal{C}$, then

$$A^2 = 12I + 5A + 6(J - I - A)$$

so: $\{A, I, J-I-A\}$ is a basis for $\mathcal{A}(\mathcal{R}_\Gamma)$ Dimension 3 over $\mathbb{C}$

RECALL:

Automorphisms of Γ $\mathrm{Aut}(\Gamma)$ $\rightsquigarrow$ Commutative C^*-algebra $C(\mathrm{Aut}(\Gamma))$ $\xrightarrow{\text{Liberation}}$ Quantum automorphism $C(\mathrm{Aut}(\Gamma)^+)$, $\mathrm{Aut}(\Gamma)^+$

magic unitary : $U = (u_{ij})$ element in some C^*-algebra $(n\times n)$

(1) Each u_{ij} is a projection: $u_{ij}^2 = u_{ij}^* = u_{ij}$

(2) "Partition of unity":

$$\sum_{l=1}^{n} u_{il} = \sum_{l=1}^{n} u_{li} = 1,$$

$$u_{ij}u_{ik} = \delta_{jk} u_{ij}, \qquad u_{ji}u_{ki} = \delta_{jk} u_{ji}$$

((3) U is unitary: $UU^* = I = U^*U$)

Quantum permutation group :

$$S_n^+ \longleftrightarrow C(S_n^+) := \langle u_{ij} : (1) \ \& \ (2) \text{ above} \rangle$$

If Γ has $n\times n$ adj matrix A,

quantum automorphism gp $\mathrm{Aut}(\Gamma)^+$:

$$\mathrm{Aut}(\Gamma)^+ \longleftrightarrow C(\mathrm{Aut}(\Gamma^+)) := C(S_n^+)/\langle AU = UA \rangle$$

$$\left(C(\mathrm{Aut}(\Gamma)) \cong C(\mathrm{Aut}(\Gamma)^+) / \langle ab = ba \rangle \right)$$

<u>RELATIONS</u>: If α & γ are adjacent in Γ and β & δ are <u>nonadjacent</u>, then

$$u_{\alpha\beta} u_{\gamma\delta} = u_{\gamma\delta} u_{\alpha\beta} = u_{\beta\alpha} u_{\delta\gamma} = u_{\delta\gamma} u_{\beta\alpha} = 0$$

<u>NEW IDEA</u>: (Lupini, Mančinska, Roberson 2020)

Define an equivalence relation $\sim_2$ on $V(\Gamma) \times V(\Gamma)$ by:

$$(\alpha, \beta) \sim_2 (\gamma, \delta) \quad \text{if} \quad u_{\alpha\gamma} u_{\beta\delta} \neq 0$$

<u>Equivalence classes</u>: <u>Quantum orbitals</u>

<u>THM</u> (LMR 2020)

The quantum orbitals form a CC $\mathcal{R}_{\mathrm{Aut}(\Gamma)^+}$ w/ corresponding coherent algebra $\mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\Gamma)^+})$.

<u>IDEA</u>: $\mathcal{A}(\mathcal{R}_\Gamma) \subseteq \mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\Gamma)^+}) \subseteq \mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\Gamma)})$

<u>THM</u> (LMR 2020) $u_{\alpha\beta} u_{\gamma\delta} = 0$ <u>unless</u> (α,γ), (β,δ) are both in the <u>same</u> class of $\mathcal{R}_{\mathrm{Aut}(\Gamma)^+}$

<u>THM</u> (LMR 2020) Almost all graphs on n vertices have trivial quantum automorphism gp as $n \longrightarrow \infty$.

<u>Pf</u>

- Babai, Kucera (1979): Almost all graphs on n vertices have <u>trivial</u> coherent configuration $\mathcal{R}_\Gamma$, i.e.,

$$\mathcal{R}_\Gamma = \{ \{(\alpha,\beta)\} : \alpha, \beta \in V(\Gamma)\}$$

- Thus $M_n(\mathbb{C}) = \mathcal{A}(\mathcal{R}_\Gamma) \subseteq \mathcal{A}(\mathcal{R}_{\mathrm{Aut}(\Gamma)^+})$, so $\mathcal{O}(\mathcal{R}_{\mathrm{Aut}(\Gamma)^+}) = M_n(\mathbb{C})$

<u>and</u> $\mathrm{Aut}(\Gamma) = 1$

- So, unless $\alpha = \gamma$ & $\beta = \delta$,

  $$u_{\alpha\beta} u_{\gamma\delta} = 0;$$

  otherwise, $u_{\alpha\beta} u_{\alpha\beta} = u_{\alpha\beta}$

- So: $C(\mathrm{Aut}(\Gamma)^+)$ is commutative!

  and so "$\mathrm{Aut}(\Gamma)^+ =$" $\mathrm{Aut}(\Gamma) = 1$

$\square$