Introduction to Quantum Groups

III. Quantum automorphisms of graphs

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Oct. 21, 2020

Introduction to Quantum Groups Series

- Sep. 28: *Groups, algebras and duality*
  P. Clare
- Oct. 7: *q-Deformations of Lie algebras*
  E. Shelburne'21
- Oct. 21: *Quantum automorphisms of graphs*
  S. Phillips'21 and A. Pisharody'21
- Oct. 28: *Quantum automorphism groups of finite graphs: a survey*
  M. Weber
- Nov. 11: *TBD*
  E. Swartz

Today's objectives

1. Generic instance of liberation: quantum groups à la Woronowicz.

Compact group		Commutative C*-algebra		Woronowicz algebra
G	$\rightsquigarrow$	$C(G)$	$\rightsquigarrow$	$\mathcal{A}$ (Matrix quantum group)

Permutation Group		Commutative C*-algebra		Quantum Permutation Group
S_n	$\rightsquigarrow$	$C(S_n)$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(S_n^+)$

2. Characterize classical permutation groups with a Woronowicz algebra and use the process of liberation introduced in the previous discussion of compact matrix quantum groups to define quantum permutation groups.

3. Apply the above to automorphisms of finite graphs to define quantum automorphisms.

Permutations as matrices

S_n is the group consisting of bijections of a set with n elements.

To any permutation $\sigma \in S_n$ one can associate a matrix of size $n \times n$:

$$\tilde{\sigma} := \left[\begin{array}{c|c|c|c|c} e_{\sigma(1)} & \cdots & e_{\sigma(k)} & \cdots & e_{\sigma(n)} \end{array} \right]$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{C}^n$.

Example: the permutations (12) and (132) in S_3 are represented by

$$\widetilde{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widetilde{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Permutations as Matrices

General properties:

- All permutation matrices $\tilde{\sigma}$ are *unitary* $(\tilde{\sigma}^*\tilde{\sigma} = I_n)$.
- The map $\sigma \longmapsto \tilde{\sigma}$ is an injective group homomorphism.

$\rightsquigarrow$ S_n can be viewed as a subgroup of the unitary group: $\boxed{S_n \leq \mathrm{U}(n)}$.

Observation:

The matrices coming from S_n are not just unitary... they are *magic!*

Recall that a matrix in $M_n(\mathbb{C})$ is called *magic* if its a square matrix, all of whose entries are projections, 0 and 1 in $\mathbb{C}$, and all of whose rows and columns are partitions of unity, meaning the projections are pairwise orthogonal and their sums equal 1.

The notion of magic unitary generalizes to matrices with entries in a C*-algebra and plays a central role in the definition of quantum permutations.

S_n as a Quantum Group: the Generating Matrix

Define

$$u_{ij} : S_n \longrightarrow \mathbb{C}$$

by

$$u_{ij} : \begin{bmatrix} g_{11} & & \cdots & & g_{1n} \\ & \ddots & & & \\ \vdots & & g_{ij} & & \vdots \\ & & & \ddots & \\ g_{n1} & & \cdots & & g_{nn} \end{bmatrix} \longmapsto g_{ij}$$

Fact:

Then $C(S_n)$ is generated by these u_{ij}'s.

Idea: think of this generating family as the entries in a matrix $u = [u_{ij}] \in M_n(C(S_n))$.

S_n as a Quantum Group: the Generating Matrix

Observations:

- All the entries of u are *projections* in $C(S_n)$: $u_{ij}^2 = u_{ij} = u_{ij}^*$
- $u_{ij}u_{ik} = \delta_{jk}u_{ij}$ and $u_{ji}u_{ki} = \delta_{jk}u_{ji}$ for $1 \leq i, j \leq n$
- $\sum_{l=1}^{n} u_{il} = 1 = \sum_{l=1}^{n} u_{li}$ for $1 \leq i \leq n$
- u is *unitary* in $M_n(C(S_n))$.

In other words, the matrix u is a *magic unitary* in $M_n(C(S_n))$.

Conclusion

The C^*-algebra $C(S_n)$ is the universal commutative C^*-algebra generated by n^2 elements $\{u_{ij}, 1 \leq i, j \leq n\}$, with relations making $u = \{u_{ij}\} \in M_n(C(S_n))$ into a magic unitary matrix.
We write

$$C(S_n) = C^*_{\text{comm}}(u_{ij} \mid u = n \times n \text{ magic unitary}).$$

S_n as a Quantum Group: Woronowicz Algebra Structure

From a previous talk in the Quantum Groups Series, we know the commutative C^*−algebra $C(S_n)$ has a structure of Woronowicz algebra:

- $C(S_n)$ is generated by n^2 elements $\{u_{ij}, 1 \leq i, j \leq n\}$.
- They form a biunitary matrix $u = [u_{ij}]$ in $M_n(C(S_n))$.
- $\Delta(u_{ij}) = \Sigma u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : C(S_n) \longrightarrow C(S_n) \otimes C(S_n)$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : C(S_n) \longrightarrow \mathbb{C}$
- $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : C(S_n) \longrightarrow C(S_n)^{\mathrm{op}}$

Conclusion:

Therefore, $C(S_n)$ is a Woronowicz algebra. We have now characterized $C(S_n)$ and are ready to modify this structure to get a non-classical object: quantum permutation groups.

Liberation of $C(S_n)$

The idea of *liberation* is to drop the commutativity condition. In other terms, $C(S_n^+)$ is defined as the not necessarily commutative C*-algebra satisfying (M.U.). We write

$$C(S_n^+) = C^*(u_{ij} | u = n \times n \text{ magic unitary}).$$

It follows that $C(S_n)$ is

$$C(S_n) \simeq C(S_n^+) / \langle ab = ba \rangle$$

where $\langle ab = ba \rangle = \left\langle ab - ba \, , \; a, b \in C(S_n^+) \right\rangle$ is the ideal.
Although there is no such object as S_n^+, we consider S_n as a subgroup of the *quantum group* S_n^+ and write

$$S_n \subseteq S_n^+ \tag{1}$$

to express that we think of the surjection

$$C(S_n^+) \longrightarrow C(S_n^+) / \langle ab = ba \rangle \simeq C(S_n) \tag{2}$$

as the restriction of functions from S_n^+ to S_n.

Quantum Permutations

We now claim that the corresponding compact quantum group $C(S_n^+)$ consists of "quantum permutations". Recall:

$$C(S_n) = C^*_{\text{comm}}(u_{ij} \mid u = n \times n \quad \text{magic unitary})$$

$$C(S_n^+) = C^*(u_{ij} \mid u = n \times n \quad \text{magic unitary})$$

We will use algebras to compare S_n and S_n^+. To do so, we will interpret the surjection

$$C(S_n^+) \longrightarrow C(S_n^+)/\langle ab = ba \rangle \simeq C(S_n)$$

to imply $S_n \subseteq S_n^+$.

Conclusion:

$C(S_n) \simeq C(S_n^+)$ if and only if $C(S_n^+)$ is commutative.

Consider S_1. By construction, S_1 contains only one element, the identity. Clearly, $C(S_1^+)$ is commutative. Then when we relax the commutativity requirement $C(S_1) \simeq C(S_1^+)$.

Now consider S_2. Then $C(S_2)$ is generated by 2 elements, $p, 1-p$. We have a magic unitary of the form

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Then p and $1-p$ automatically commute and so $C(S_2^+)$ is commutative. Therefore, $C(S_2) \simeq C(S_2^+)$.

Quantum Permutations

Proposition (Lupini et al., 2020).

$C(S_3^+)$ is commutative.

Consider S_3.

- Suppose $u = [u_{ij}]_{i,j\in[3]}$ is a magic unitary. By the definition of magic unitaries, all pairs u_{ij} and u_{lk} in the same row ($i = l$) and/or column ($j = k$) commute. So, we only need to show that u_{ij} and u_{lk} commute for $i \neq l$ and $j \neq k$. Since we can permute the rows and columns of a magic unitary independently and always have a magic unitary, it suffices to show that $u_{11}u_{22} = u_{22}u_{11}$.
- We have that

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{12} + u_{11}u_{22}u_{13} && u_{22}u_{12} = 0 \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \end{aligned}$$

- Additionally,

$$\begin{aligned} u_{11}u_{22}u_{13} &= u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{13} - u_{11}u_{21}u_{13} - u_{11}u_{23}u_{13} && u_{11}u_{21} = 0, u_{23}u_{13} = 0 \\ &= u_{11}u_{13} \\ &= 0 \end{aligned}$$

- Therefore, $u_{11}u_{22} = u_{11}u_{22}u_{11}$ and thus, $u_{22}u_{11} = u_{11}u_{22}$ by applying $*$ to this equation.

Then we know $C(S_3^+)$ is commutative and $C(S_3) \simeq C(S_3^+)$

Quantum Permutations

Consider S_4. There exists non-commuting projections p and q on the Hilbert space $\mathcal{H}$ that generate an infinite-dimensional sub-C^*-algebra of $B(\mathcal{H})$. Then $C(S_4^+)$ surjects onto

$$\begin{bmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$$

Observe that this matrix is a magic unitary, so $C(S_4^+)$ has a noncommutative quotient, and $C(S_4^+)$ is not commutative as well. Therefore, $C(S_4^+)$ is not isomorphic to $C(S_4)$.
Then "S_4^+" consists of quantum permutations.

More generally:

For $n \geq 4$, $C(S_n^+)$ is not commutative, and infinite dimensional. In particular:

$$C(S_n) \not\cong C(S_n^+).$$

Conclusion:

Quantum permutations start existing at $n = 4$.

Overview So Far:

What We've Shown So Far:

Permutation Group		Commutative C*-algebra		Quantum Permutation Group
S_n	$\rightsquigarrow$	$C(S_n)$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(S_n^+)$

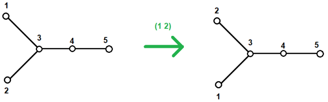
Goal: Apply this to Automorphisms of Graphs

Automorphisms of Γ		Commutative C*-algebra		Quantum Automorphisms
$\mathrm{Aut}(\Gamma)$	$\rightsquigarrow$	$C(\mathrm{Aut}(\Gamma))$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(\mathrm{Aut}(\Gamma)^+)$

What are Graph Automorphisms

- Let Γ be a graph on n vertices. A graph automorphism is a bijection $\sigma : V(\Gamma) \to V(\Gamma)$ that preserves adjacency and non-adjacency between vertices.

  Example: For the graph Γ below, the transposition $\sigma = (1\ 2)$ is an automorphism:

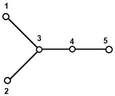


- The automorphisms of Γ form a group, denoted by $\mathrm{Aut}(\Gamma)$.
  In the example, $\mathrm{Aut}(\Gamma) = \{e, (1\ 2)\}$

$\mathrm{Aut}(\Gamma) \leq S_n$, so we can also consider automorphisms as as matrices

Which matrices are automorphisms of Γ?

- The adjacency matrix of a graph Γ with n vertices is the $n \times n$ matrix ε where ε_{ij} is the number of edges between vertices i and j.
  Example:



$$\varepsilon = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- For the example above, a quick calculation shows that the automorphims σ and e commute with the adjacency matrix of Γ

$$\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Automorphisms of a graph Γ commute with the adjacency matrix of Γ. Specifically, for $\sigma \in S_n$,

$$\sigma \in \mathrm{Aut}(\Gamma) \Leftrightarrow \sigma\varepsilon = \varepsilon\sigma$$

Proof that Graph Automorphisms Commute with the Adjacency Matrix

Let $\sigma \in S_n$. Then $\sigma \in \mathrm{Aut}(\Gamma) \iff \sigma\varepsilon = \varepsilon\sigma$.

- Let $\sigma \in S_n$, and $\sigma \in \mathrm{Aut}(\Gamma)$
- As functions,

$$\sigma : e_i \longmapsto e_{\sigma(i)} \qquad \varepsilon : e_i \longmapsto \sum_{i\sim j} e_j$$

$$\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

  where $i \sim j$ means i and j are related by an edge.
- Composing them,

$$\varepsilon \circ \sigma : e_i \longmapsto \sum_{\sigma(i)\sim j} e_j$$

$$\sigma \circ \varepsilon : e_i \longmapsto \sum_{i\sim j} e_{\sigma(j)}$$

- Since $\sigma \in \mathrm{Aut}(\Gamma)$

$$\sigma(i) \sim j \iff j = \sigma(k), \text{or some } k \text{ where } i \sim k$$

- Therefore

$$\varepsilon \circ \sigma(e_i) = \sum_{\sigma(i)\sim j} e_j = \sum_{i\sim k} e_{\sigma(k)} = \sigma \circ \varepsilon(e_i)$$

- The converse is shown in a similar way.

How do we characterize $C(\mathrm{Aut}(\Gamma))$?

- We have seen that $C(S_n)$ is the universal commutative C^*-algebra:
$$C(S_n) := C^*_{\mathrm{comm}}(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_\ell u_{i\ell} = \sum_\ell u_{\ell i} = 1)$$

- Similarly, $C(\mathrm{Aut}(\Gamma))$ is the universal commutative C^*-algebra:
$$C(\mathrm{Aut}(\Gamma)) := C^*_{\mathrm{comm}}(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_\ell u_{i\ell} = \sum_\ell u_{\ell i} = 1, u\varepsilon = \varepsilon u)$$

- The inclusion of groups $\mathrm{Aut}(\Gamma) \le S_n$ is reflected by a surjection of C*-algebras $C(S_n) \to C(\mathrm{Aut}(\Gamma))$. Indeed:
$$C(\mathrm{Aut}(\Gamma)) \cong C(S_n)/\langle \varepsilon u = u\varepsilon \rangle$$

How do we liberate $C(\mathrm{Aut}(\Gamma))$?

- Recall that $C(S_n^+)$ is the universal not necessarily commutative C^*-algebra:
$$C(S_n^+) := (u_{ij}, 1 \leq i,j \leq n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_\ell u_{i\ell} = \sum_\ell u_{\ell i} = 1)$$

- (Banica, 2005) Similarly we may define the quantum automorphism group $\mathrm{Aut}(\Gamma)^+$ based on the Woronowicz algebra:
$$C(\mathrm{Aut}(\Gamma)^+) := C(S_n^+)/\langle \varepsilon u = u\varepsilon \rangle$$

- As before with the quantum permuation group,
$$C(\mathrm{Aut}(\Gamma)^+)/\langle ab = ba \rangle \cong C(\mathrm{Aut}(\Gamma))$$

  There is a surjection $C(\mathrm{Aut}(\Gamma)^+) \longrightarrow C(\mathrm{Aut}(\Gamma))$, so we can say:

$$\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Gamma)^+$$

What does it mean to have quantum symmetry?

If the algebra $C(\mathrm{Aut}(\Gamma)^+)$ is commutative:

- $C(\mathrm{Aut}(\Gamma)^+) := C(S_n+)/\langle \varepsilon u = u\varepsilon \rangle$ is the same as $C(\mathrm{Aut}(\Gamma))$
- Not only is there a surjection $C(\mathrm{Aut}(\Gamma)^+) \longrightarrow C(\mathrm{Aut}(\Gamma))$, there is a bijection
- $C(\mathrm{Aut}(\Gamma)^+) \cong C(\mathrm{Aut}(\Gamma))$, so we say a graph has no quantum symmetry, or

$$\mathrm{Aut}(\Gamma) \overset{\text{👻}}{=} \mathrm{Aut}(\Gamma)^+$$

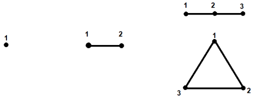
If the algebra $C(\mathrm{Aut}(\Gamma)^+)$ is NOT commutative:

- There is a surjection from $C(\mathrm{Aut}(\Gamma)^+)$ to $C(\mathrm{Aut}(\Gamma))$, but no bijection
- In some sense there are "extra elements" in $\mathrm{Aut}(\Gamma)^+$
- We can say a graph has quantum symmetry, or there are "quantum automorphims," and

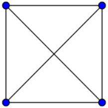
$$\mathrm{Aut}(\Gamma) \overset{\text{👻}}{<} \mathrm{Aut}(\Gamma)^+$$

Graphs with and without Quantum Symmetry

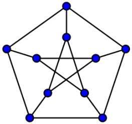
- As we had before with $n = 1, 2, 3$, we know $C(S_n) \cong C(S_n^+)$.
  So $\mathrm{Aut}(\Gamma)^+ = Aut(\Gamma)$ for these graphs, they have no quantum symmetry.



- K_4 the complete graph on 4 vertices has quantum symmetry. In fact, $\mathrm{Aut}(K_4)^+ = S_4^+$ (Banica-Bichon, 2007).



- The Petersen Graph has no quantum symmetry, $\mathrm{Aut}(P)^+ = \mathrm{Aut}(P) = S_5$ (Schmidt, 2018).



Wrapping Up

Overall View:

Compact group		Commutative C*-algebra		Woronowicz algebra
G	$\rightsquigarrow$	$C(G)$	$\overset{\text{liberation}}{\rightsquigarrow}$	$\mathcal{A}$ (Matrix quantum group)

First Part of Today's Talk:

Permutation Group		Commutative C*-algebra		Quantum Permutation Group
S_n	$\rightsquigarrow$	$C(S_n)$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(S_n^+)$

Second Part of Today's Talk:

Automorphisms of Γ		Commutative C*-algebra		Quantum Automorphisms
$\mathrm{Aut}(\Gamma)$	$\rightsquigarrow$	$C(\mathrm{Aut}(\Gamma))$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(\mathrm{Aut}(\Gamma)^+)$

Thank you

References

- T. Banica, J. Bichon and B. Collins. "Quantum permutation groups: a survey." Noncommutative harmonic analysis with applications to probability, Banach Center Publ., 78, Polish Acad. Sci. Inst. Math., Warsaw, (2007): 13–34.

- T. Banica and J. Bichon. "Quantum automorphism groups of vertex-transitive graphs of order $\leq$ 11." Journal of Algebraic Combinatorics 26.1 (2007): 83.

- M. Lupini, L. Mančinska and D. E. Roberson. "Nonlocal games and quantum permutation groups." Journal of Functional Analysis (2020): 108592.

- S. Schmidt. "The Petersen graph has no quantum symmetry." Bulletin of the London Mathematical Society 50.3 (2018): 395–400.