Orthogonal Units of the Trivial Source Ring

Rob Carman

W&M GAG Seminar

October 14, 2020

Notation

- G: a finite group
- p: a prime number
- F: an algebraically closed field of characteristic p.
- S: a Sylow p-subgroup of G
- $\mathcal{F}_S(G)$: the fusion system on S induced by G
- FG: the group algebra

Green Ring

The isomorphism classes of finitely generated FG-modules form a commutative semiring with

$$[V] + [W] = [V \oplus W] \quad \text{and} \quad [V][W] = [V \otimes_F W]$$

where $V \otimes_F W$ has diagonal G-action: $g(v \otimes w) = (gv) \otimes (gw)$. The identity is $[F]$, the class of the trivial FG-module.

Green Ring

The isomorphism classes of finitely generated FG-modules form a commutative semiring with

$$[V] + [W] = [V \oplus W] \quad \text{and} \quad [V][W] = [V \otimes_F W]$$

where $V \otimes_F W$ has diagonal G-action: $g(v \otimes w) = (gv) \otimes (gw)$. The identity is $[F]$, the class of the trivial FG-module. The Grothendieck ring of this semiring is called the Green ring, denoted $A_F(G)$. It is free abelian with basis given by the indecomposable FG-modules, but there are infinitely many in general.

Brauer Character Ring

Let I be the subgroup of $A_F(G)$ generated by elements of the form $[M] - [L] - [N]$ where

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of FG-modules.

Brauer Character Ring

Let I be the subgroup of $A_F(G)$ generated by elements of the form $[M] - [L] - [N]$ where

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of FG-modules. Since tensoring over the field F is an exact functor, I is an ideal of $A_F(G)$. The quotient ring $R_F(G) = A_F(G)/I$ is called the Brauer character ring.

Brauer Character Ring

Let I be the subgroup of $A_F(G)$ generated by elements of the form $[M] - [L] - [N]$ where

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of FG-modules. Since tensoring over the field F is an exact functor, I is an ideal of $A_F(G)$. The quotient ring $R_F(G) = A_F(G)/I$ is called the Brauer character ring. In $R_F(G)$, we still have $[V] + [W] = [V \oplus W]$ and $[V][W] = [V \otimes_F W]$, but there are many more relations, and $R_F(G)$ now has a basis given by the irreducible modules, of which there are finitely many.

Burnside Ring

The Burnside ring $B(G)$ is the Grothendieck ring of the category of finite (left) G-sets. For G-sets X and Y:

$$[X] + [Y] = [X \sqcup Y] \quad \text{and} \quad [X][Y] = [X \times Y]$$

Burnside Ring

The Burnside ring $B(G)$ is the Grothendieck ring of the category of finite (left) G-sets. For G-sets X and Y:

$$[X] + [Y] = [X \sqcup Y] \quad \text{and} \quad [X][Y] = [X \times Y]$$

$B(G)$ is also free abelian, with basis given by isomorphism classes of the transitive G-sets:

Burnside Ring

The Burnside ring $B(G)$ is the Grothendieck ring of the category of finite (left) G-sets. For G-sets X and Y:

$$[X] + [Y] = [X \sqcup Y] \quad \text{and} \quad [X][Y] = [X \times Y]$$

$B(G)$ is also free abelian, with basis given by isomorphism classes of the transitive G-sets: $[G/H]$ as H runs over representatives of the conjugacy classes of subgroups of G.

Ghost Ring

For any G-set X and any $H \leq G$, we can count the number of H-fixed points of X. In this way, we can define a ring morphism $B(G) \to \mathbb{Z}, [X] \mapsto |X^H|$.

Ghost Ring

For any G-set X and any $H \leq G$, we can count the number of H-fixed points of X. In this way, we can define a ring morphism $B(G) \to \mathbb{Z}, [X] \mapsto |X^H|$. If $H, K \leq G$ are conjugate, then $|X^H| = |X^K|$. We can therefore define

$$B(G) \longrightarrow \left(\prod_{H \leq G} \mathbb{Z} \right)^G, \quad [X] \mapsto (|X^H|)_{H \leq G}.$$

called the *ghost map* for $B(G)$.

Ghost Ring

For any G-set X and any $H \leq G$, we can count the number of H-fixed points of X. In this way, we can define a ring morphism $B(G) \to \mathbb{Z}, [X] \mapsto |X^H|$. If $H, K \leq G$ are conjugate, then $|X^H| = |X^K|$. We can therefore define

$$B(G) \longrightarrow \left(\prod_{H \leq G} \mathbb{Z} \right)^G, \quad [X] \mapsto (|X^H|)_{H \leq G}.$$

called the *ghost map* for $B(G)$. The ghost map is an injective ring morphism with finite cokernel.

Units of the Burnside Ring

In particular, $B(G)^\times \leq (\prod_{H \leq G} \{\pm 1\})^G$, hence an elementary abelian 2-group, i.e. an $\mathbb{F}_2$-vector space.

Units of the Burnside Ring

In particular, $B(G)^\times \leq (\prod_{H \leq G}\{\pm 1\})^G$, hence an elementary abelian 2-group, i.e. an $\mathbb{F}_2$-vector space. In general the dimension of $B(G)^\times$ is unknown, but since S is a p-group, Serge Bouc's theory of biset functors can be used to find a basis of $B(S)^\times$.

Units of the Burnside Ring

In particular, $B(G)^\times \leq (\prod_{H \leq G} \{\pm 1\})^G$, hence an elementary abelian 2-group, i.e. an $\mathbb{F}_2$-vector space. In general the dimension of $B(G)^\times$ is unknown, but since S is a p-group, Serge Bouc's theory of biset functors can be used to find a basis of $B(S)^\times$. And we know that the image of $\mathrm{Res}^G_S : B(G) \to B(S)$ is $B(\mathcal{F}_S(G))$.

Units of the Burnside Ring

In particular, $B(G)^\times \leq (\prod_{H\leq G}\{\pm 1\})^G$, hence an elementary abelian 2-group, i.e. an $\mathbb{F}_2$-vector space. In general the dimension of $B(G)^\times$ is unknown, but since S is a p-group, Serge Bouc's theory of biset functors can be used to find a basis of $B(S)^\times$. And we know that the image of $\mathrm{Res}^G_S : B(G) \to B(S)$ is $B(\mathcal{F}_S(G))$. If $a \in B(S)$, then $a \in B(\mathcal{F}_S(G))$ iff $|a^P| = |a^Q|$ for all $P, Q \leq S$ that are conjugate in G.

Units of the Burnside Ring

In particular, $B(G)^\times \leq (\prod_{H\leq G}\{\pm 1\})^G$, hence an elementary abelian 2-group, i.e. an $\mathbb{F}_2$-vector space. In general the dimension of $B(G)^\times$ is unknown, but since S is a p-group, Serge Bouc's theory of biset functors can be used to find a basis of $B(S)^\times$. And we know that the image of $\mathrm{Res}^G_S : B(G) \to B(S)$ is $B(\mathcal{F}_S(G))$. If $a \in B(S)$, then $a \in B(\mathcal{F}_S(G))$ iff $|a^P| = |a^Q|$ for all $P, Q \leq S$ that are conjugate in G. Moreover, there exists a ring map $t^G_S : B(\mathcal{F}_S(G)) \to B(G)$ such that $\mathrm{Res}^G_S \circ t^G_S$ is the identity.

Trivial Source Ring

A *trivial source* (aka *p-permutation*) FG-module M is a direct summand of a permutation FG-module: there is an FG-module M' and a G-set X such that $M \oplus M' \cong FX$.

Trivial Source Ring

A *trivial source* (aka *p-permutation*) FG-module M is a direct summand of a permutation FG-module: there is an FG-module M' and a G-set X such that $M \oplus M' \cong FX$. If M and N are two trivial source FG-modules, then $M \oplus N, M \otimes_F N$, and $M^\circ = \mathrm{Hom}_F(M, F)$ are also trivial source FG-modules.

Trivial Source Ring

A *trivial source* (aka *p-permutation*) FG-module M is a direct summand of a permutation FG-module: there is an FG-module M' and a G-set X such that $M \oplus M' \cong FX$. If M and N are two trivial source FG-modules, then $M \oplus N, M \otimes_F N$, and $M^\circ = \mathrm{Hom}_F(M, F)$ are also trivial source FG-modules. The *trivial source ring* $T_F(G)$ is the Grothendieck ring of the isomorphism classes of trivial source FG-modules. The operations are given by

$$[M] + [N] = [M \oplus N] \quad \text{and} \quad [M][N] = [M \otimes_F N].$$

Trivial Source Ring

$T_F(G)$ is free abelian with basis given by the indecomposable trivial source modules, of which there are finitely many, though they can be difficult to determine.

Trivial Source Ring

$T_F(G)$ is free abelian with basis given by the indecomposable trivial source modules, of which there are finitely many, though they can be difficult to determine.

We have the linearization map $l_G : B(G) \to T_F(G), [X] \mapsto [FX]$. This is an isomorphism when G is a p-group.

Torsion Unit Groups

We consider the torsion subgroup of the unit group of each of the last three rings. $B(G)^\times$ is a finite group.

Torsion Unit Groups

We consider the torsion subgroup of the unit group of each of the last three rings. $B(G)^\times$ is a finite group. We denote these groups as $U_\circ(R_F(G))$ and $U_\circ(T_F(G))$ for the other two rings since inverses of such units must come from their dual modules.

Torsion Unit Groups

We consider the torsion subgroup of the unit group of each of the last three rings. $B(G)^\times$ is a finite group. We denote these groups as $U_\circ(R_F(G))$ and $U_\circ(T_F(G))$ for the other two rings since inverses of such units must come from their dual modules.

Theorem: $U_\circ(R_F(G)) = \{\pm[V] : \dim_F(V) = 1\} \cong \{\pm 1\} \times \mathrm{Hom}(G, F^\times)$.

Torsion Unit Groups

We consider the torsion subgroup of the unit group of each of the last three rings. $B(G)^\times$ is a finite group. We denote these groups as $U_\circ(R_F(G))$ and $U_\circ(T_F(G))$ for the other two rings since inverses of such units must come from their dual modules.

Theorem: $U_\circ(R_F(G)) = \{\pm[V] : \dim_F(V) = 1\} \cong \{\pm 1\} \times \mathrm{Hom}(G, F^\times)$.

Goal: Determine $U_\circ(T_F(G))$ for any group G.

Motivation: Elements of $U_\circ(T_F(G))$ determine p-permutation auto-equivalences of blocks of FG.

Brauer Construction

Let M be a trivial source FG-module and P be a p-subgroup of G.

Then the Brauer construction

$$M(P) := M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q)$$

is an $F[N_G(P)/P]$-module, where $\mathrm{tr}_Q^P : M^Q \to M^P, m \mapsto \sum_{a\in P/Q} am$ is the trace map.

Brauer Construction

Let M be a trivial source FG-module and P be a p-subgroup of G.

Then the Brauer construction

$$M(P) := M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q)$$

is an $F[N_G(P)/P]$-module, where $\mathrm{tr}_Q^P : M^Q \to M^P, m \mapsto \sum_{a \in P/Q} am$ is the trace map. We get a ring morphism $T_F(G) \to R_F(N_G(P)/P), [M] \mapsto [M(P)]$.

Brauer Construction

Let M be a trivial source FG-module and P be a p-subgroup of G.

Then the Brauer construction

$$M(P) := M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q)$$

is an $F[N_G(P)/P]$-module, where $\mathrm{tr}_Q^P : M^Q \to M^P, m \mapsto \sum_{a\in P/Q} am$

is the trace map. We get a ring morphism

$T_F(G) \to R_F(N_G(P)/P), [M] \mapsto [M(P)]$. And $[M] = [N]$ in $T_F(G)$ iff

$[M(P)] = [N(P)]$ for all p-subgroups of P (only need to check up to

G-conjugation).

Determining $U_\circ(T_F(G))$

We have an injective ring map

$$T_F(G) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} R_F(N_G(P)/P)$$

Determining $U_\circ(T_F(G))$

We have an injective ring map

$$T_F(G) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} R_F(N_G(P)/P)$$

Hence an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} U_\circ(R_F(N_G(P)/P)).$$

Determining $U_\circ(T_F(G))$

We have an injective ring map

$$T_F(G) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} R_F(N_G(P)/P)$$

Hence an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} U_\circ(R_F(N_G(P)/P)).$$

But we know the orthogonal units for each $R_F(N_G(P)/P)$:

$$U_\circ(T_F(G)) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} \{\pm 1\} \times \mathrm{Hom}(N_G(P)/P, F^\times)$$

Determining $U_\circ(T_F(G))$

We have an injective ring map

$$T_F(G) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} R_F(N_G(P)/P)$$

Hence an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} U_\circ(R_F(N_G(P)/P)).$$

But we know the orthogonal units for each $R_F(N_G(P)/P)$:

$$U_\circ(T_F(G)) \longrightarrow \prod_{[P]\in\mathcal{F}_S(G)} \{\pm 1\} \times \mathrm{Hom}(N_G(P)/P, F^\times)$$

The tuple of signs correspond to a unit of the Burnside ring of the fusion system.

Determining $U_\circ(T_F(G))$

So we have an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow B(\mathcal{F}_S(G))^\times \times \prod_{[P]\in\mathcal{F}_S(G)} \mathrm{Hom}(N_G(P)/P, F^\times).$$

Determining $U_\circ(T_F(G))$

So we have an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow B(\mathcal{F}_S(G))^\times \times \prod_{[P]\in\mathcal{F}_S(G)} \mathrm{Hom}(N_G(P)/P, F^\times).$$

By using the t_S^G map, we see that this projects onto the first component $B(\mathcal{F}_S(G))^\times$,

Determining $U_\circ(T_F(G))$

So we have an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow B(\mathcal{F}_S(G))^\times \times \prod_{[P] \in \mathcal{F}_S(G)} \mathrm{Hom}(N_G(P)/P, F^\times).$$

By using the t_S^G map, we see that this projects onto the first component $B(\mathcal{F}_S(G))^\times$, which can be determined by using the biset functor theory and some group actions determined by the automorphism groups of the fusion system.

Determining $U_\circ(T_F(G))$

So we have an injective group homomorphism

$$U_\circ(T_F(G)) \longrightarrow B(\mathcal{F}_S(G))^\times \times \prod_{[P]\in\mathcal{F}_S(G)} \mathrm{Hom}(N_G(P)/P, F^\times).$$

By using the t_S^G map, we see that this projects onto the first component $B(\mathcal{F}_S(G))^\times$, which can be determined by using the biset functor theory and some group actions determined by the automorphism groups of the fusion system. So we are left to determine which homs show up here. They seemed to be "glued" together in some way. (via transporters?)

Determining $U_\circ(T_F(G))$

When $G = S \times H$, we have $T_F(G) = B(S) \otimes R_F(H)$ and $U_\circ(T_F(G)) = B(S)^\times \times \mathrm{Hom}(H, F^\times)$, so we only get the "bottom" component corresponding to the trivial p-subgroup.

Determining $U_\circ(T_F(G))$

When $G = S \times H$, we have $T_F(G) = B(S) \otimes R_F(H)$ and $U_\circ(T_F(G)) = B(S)^\times \times \mathrm{Hom}(H, F^\times)$, so we only get the "bottom" component corresponding to the trivial p-subgroup.

For $G = C_3 \rtimes D_8, p = 2$ there exists $1 < P < S$ such that $\mathrm{Hom}(N_G(P)/P, F^\times) \cong C_3$, but there is no $u \in U_\circ(T_F(G))$ such that $u(P)$ is nontrivial.

Determining $U_\circ(T_F(G))$

When $G = S \times H$, we have $T_F(G) = B(S) \otimes R_F(H)$ and $U_\circ(T_F(G)) = B(S)^\times \times \mathrm{Hom}(H, F^\times)$, so we only get the "bottom" component corresponding to the trivial p-subgroup.

For $G = C_3 \rtimes D_8, p = 2$ there exists $1 < P < S$ such that $\mathrm{Hom}(N_G(P)/P, F^\times) \cong C_3$, but there is no $u \in U_\circ(T_F(G))$ such that $u(P)$ is nontrivial.

For $G = A_4, p = 2$, we get every hom independently, i.e. the map is an isomorphism.

Semidirect Products

Suppose $S \trianglelefteq G$. Then by the Schur-Zassenhaus Theorem,

$G = SH \cong S \rtimes H$ for some p'-group H.

Semidirect Products

Suppose $S \trianglelefteq G$. Then by the Schur-Zassenhaus Theorem, $G = SH \cong S \rtimes H$ for some p'-group H. Moreover, for any $P \leq S$, we also have $N_G(P) = N_S(P)C_P$ for a p'-group C_P.

Semidirect Products

Suppose $S \trianglelefteq G$. Then by the Schur-Zassenhaus Theorem, $G = SH \cong S \rtimes H$ for some p'-group H. Moreover, for any $P \leq S$, we also have $N_G(P) = N_S(P)C_P$ for a p'-group C_P. If V is an FC_P-module, then we get an $F[PC_P]$-module $\tilde{V}$ that is V as an F-vector space, but with PC_P-action such that P acts trivially.

Semidirect Products

Suppose $S \trianglelefteq G$. Then by the Schur-Zassenhaus Theorem, $G = SH \cong S \rtimes H$ for some p'-group H. Moreover, for any $P \leq S$, we also have $N_G(P) = N_S(P)C_P$ for a p'-group C_P. If V is an FC_P-module, then we get an $F[PC_P]$-module $\tilde{V}$ that is V as an F-vector space, but with PC_P-action such that P acts trivially. Then

$$P \rtimes V := \mathrm{Ind}_{PC_P}^{G}(\tilde{V}) = FG \otimes_{F[PC_P]} \tilde{V}$$

is an FG-module.

$G = S \rtimes H$

Proposition: For any $P \leq S$ and any irreducible FC_P-module V, $P \rtimes V$ is an indecomposable trivial source FG-module, that does not depend on a choice of C_P.

$G = S \rtimes H$

Proposition: For any $P \leq S$ and any irreducible FC_P-module V, $P \rtimes V$ is an indecomposable trivial source FG-module, that does not depend on a choice of C_P.

Moreover, every indecomposable trivial source FG-module is of this form, and so the classes $[P \rtimes V]$ form a basis of $T_F(G)$, as P runs over G-conjugacy classes of subgroups of S and V runs over irreducible FC_P-modules.

Brauer Constructions for $G = S \rtimes H$

Proposition: If $P, Q \leq S$ and V is an FC_P-module, then

$$(P \rtimes V)(Q) \cong FT_G(P,Q) \otimes_{F[PC_P]} \tilde{V},$$

Brauer Constructions for $G = S \rtimes H$

Proposition: If $P, Q \leq S$ and V is an FC_P-module, then

$$(P \rtimes V)(Q) \cong FT_G(P,Q) \otimes_{F[PC_P]} \tilde{V},$$

where $T_G(P,Q)$ is the transporter

$$T_G(Q,P) = \{g \in G : {}^gP \leq Q\},$$

Brauer Constructions for $G = S \rtimes H$

Proposition: If $P, Q \leq S$ and V is an FC_P-module, then

$$(P \rtimes V)(Q) \cong FT_G(P, Q) \otimes_{F[PC_P]} \tilde{V},$$

where $T_G(P, Q)$ is the transporter

$$T_G(Q, P) = \{g \in G : {}^gP \leq Q\},$$

which is an $(N_G(Q), N_G(P))$-biset (and thus $FT_G(Q, P)$ is an $(FN_G(Q), FN_G(P))$-bimodule).

References

Barsotti, J.; Carman, R.: Burnside rings of fusion systems and their unit groups. *J. Group Theory* **23** (2020), no. 4, 709–729. MR **4116665**

Bouc, S.: The functor of units of Burnside rings for p-groups. *Comment. Math. Helv.* **82** (2007), no. 3, 583–615. MR **(2008e:19001)**

Carman, R.: Unit Groups of Representation Rings and Their Ghost Rings as Biset Functors. Thesis (Ph.D.) – University of California, Santa Cruz. *ProQuest LLC, Ann Arbor, MI*, 2017. 89 pp. ISBN: 978-0355-33336-7. MR **3768380**

Gelvin, M.; Reeh, S. P.; Yalçın, E.: On the basis of the Burnside ring of a fusion system. *J. Algebra* 423 (2015), 767–797. MR **3283739**

Yamauchi, K.: On the Units in a Character Ring. *Hokkaido Math. J.* **20** (1991), no. 3, 477–479. MR **(93a:20017)**