Introduction to Quantum Groups

II. q-Deformations of Lie Algebras

Ethan Shelburne

William & Mary

Oct. 7, 2020

An Oversimplified Look at Quantum Groups

Oversimplification

Quantum objects are given in terms of algebraic structures obtained by altering structures associated with classical objects.

$$\text{Classical Object} \overset{\text{"deformation"}}{\rightsquigarrow} \text{Quantum Object}$$

$$\text{Group} \rightsquigarrow \text{Algebra} \overset{\text{"deformation"}}{\rightsquigarrow} \text{"Altered" Algebra}$$

A Concrete Example of a Quantum Group

Goal for Today

Our goal is to discuss a concrete example of a quantum group, obtained by starting from the classical Lie group $SL(2, \mathbb{C})$ and undergoing a "deformation" process on an algebra associated with $SL(2, \mathbb{C})$.

The Lie Group $SL(2, \mathbb{C})$

Definition

A Lie group is a group G such that G is a smooth manifold and the group multiplication and inversion maps are both smooth maps.

Definition

Roughly speaking, a linear Lie group over $\mathbb{C}$ is a subgroup of $GL(n, \mathbb{C})$ defined by polynomial equations (and thus is a nice subset of $\mathbb{C}^p$).

Example

The special linear group $SL(2, \mathbb{C})$, defined as:

$$SL(2, \mathbb{C}) = \{X \in GL(2, \mathbb{C}) \mid \det(X) = 1\} \tag{1}$$

is an example of a linear Lie group.

Lie Algebras

Definition

A Lie algebra L is a vector space with a bilinear map $[,] : L \times L \to L$, called the Lie bracket, satisfying the following two conditions for $x, y, z \in L$:
(i)(*antisymmetry*):

$$[x, y] = -[y, x] \tag{2}$$

(ii)(*Jacobi Identity*)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{3}$$

A Lie subalgebra L' of a Lie algebra L is a subspace L' of L such that for any $x, y \in L'$, $[x, y] \in L'$.

Example

Let A be an associative algebra (roughly, a vector space with multiplication). Set $[a, b] = ab - ba$. Equipped with this Lie bracket, A becomes a Lie algebra, which we denote $L(A)$.

The Lie Algebra $\mathfrak{sl}(2)$

Consider the associative algebra $M_2(\mathbb{C})$. From our prior example, we can set:

$$[X, Y] = XY - YX \tag{4}$$

for all $X, Y \in M_2(\mathbb{C})$ to obtain a Lie algebra, which will denote $\mathfrak{gl}(2)$.

Now, consider the subspace:

$$\mathfrak{sl}(2) = \{X \in \mathfrak{gl}(2) \mid \mathrm{tr}(X) = 0\} \tag{5}$$

It is quick to verify that $\mathfrak{sl}(2)$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(2)$.

A Basis for $\mathfrak{sl}(2)$

We know that every traceless 2×2 matrix in $\mathfrak{sl}(2)$ is of the form:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \tag{6}$$

for all constants $x_i \in GL(2)$.

Accordingly, the following elements form a basis for $\mathfrak{sl}(2)$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{7}$$

We can quickly verify that these basis elements satisfy the following relations under the Lie bracket:

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y \tag{8}$$

How are Lie groups and Lie algebras related?

For any linear Lie group $G \subseteq GL(n)$, its associated Lie algebra is given by:

$$\{X \in M_n \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\} \tag{9}$$

In particular, $\mathfrak{sl}(2)$ is the Lie algebra of $SL(2, \mathbb{C})$.

One way to see this is through the relation:

$$\det(e^X) = e^{\mathrm{Tr}(X)} \tag{10}$$

Given some matrix X with trace zero, e^X will have determinant $e^0 = 1$.

Lie group		Lie algebra
$SL(2)$	$\rightsquigarrow$	$\mathfrak{sl}(2)$

Example

Let A be an associative algebra. Set $[a, b] = ab - ba$. Equipped with this Lie bracket, A becomes a Lie algebra, which we denote $L(A)$.

In general, given some arbitrary Lie algebra L, we do not have:

- Associativity
- A multiplicative operation
- $[x, y] = xy - yx$

Instead, we have a Lie bracket.

Tensor Algebra

For the Lie algebra L, we set:
$T^0(L) = \mathbb{C}, \qquad T^1(L) = L, \qquad T^2(L) = L \otimes L, \qquad T^3(L) = L \otimes L \otimes L \ldots$

Elements of $T^2(L)$ are either simple tensors in the form $x_1 \otimes x_2$ or sums of simple tensors.

In general:

$$T^n(L) = L^{\otimes n} = L \otimes \cdots \otimes L \text{ (n times)} \tag{11}$$

Definition

Set:

$$T(L) = \oplus_{n \geq 0} T^n(L) \tag{12}$$

and equip this space with the product defined by:

$$(x_1 \otimes \cdots \otimes x_n)(x_{n+1} \otimes \cdots \otimes x_{n+m}) = x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \otimes \cdots \otimes x_{n+m} \tag{13}$$

$T(L)$ is called the tensor algebra of L.

Since $L = T^1(L)$, we have a natural embedding of L into $T(L)$.

Let $I(L)$ be the two sided ideal of $T(L)$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$ where x and y are elements of L.

Definition

The enveloping algebra $U(L)$ of a Lie algebra L is given by:

$$U(L) = T(L)/I(L) \tag{14}$$

so that, if i_L is the composition of the natural injection of L into $T(L)$ and the canonical surjection of the tensor algebra onto the enveloping algebra, the relation:

$$i_L([x, y]) = x \otimes y - y \otimes x \tag{15}$$

is forced to hold.

Universal Property of Enveloping Algebras

Proposition

Let L be a Lie algebra and $i_L : L \to U(L)$ as before. Given any associative algebra A and any morphism of Lie algebras f from L into $L(A)$, there exists a unique morphism of algebras $\phi : U(L) \to A$ such that $\phi \circ i_L = f$.

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "i_L"'] & & A \\ & U(L) \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

The Enveloping Algebra of $\mathfrak{sl}(2)$

In particular, we will focus on the enveloping algebra of $\mathfrak{sl}(2)$, denoted $U(\mathfrak{sl}(2))$.

Proposition

$U(\mathfrak{sl}(2))$ is isomorphic to the algebra generated by the three elements X, Y, H with the three relations:

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y \tag{16}$$

Moreover, a basis for $U(\mathfrak{sl}(2))$ is given by the set $\{X^i Y^j H^k\}_{i,j,k\in\mathbb{N}}$.

Definition

Let k be a field. We define the affine plane $k[x, y]$ to be the polynomial algebra over k on variables x and y.

Proposition

We have $k[x, y]$ is a $U(\mathfrak{sl}(2))$ module with the action on the polynomial algebra $k[x, y]$ given by:

$$XP = x\frac{\partial P}{\partial y} \qquad YP = y\frac{\partial P}{\partial x} \qquad HP = x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y} \tag{17}$$

where P is some polynomial in $k[x, y]$.

In other words, we have an algebra homomorphism $\rho : U(\mathfrak{sl}(2)) \to \mathrm{End}(k[x, y])$. This homomorphism is defined on the basis:

$$\rho(X) = x\frac{\partial}{\partial y} \qquad \rho(Y) = y\frac{\partial}{\partial x} \qquad \rho(H) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{18}$$

A Quick Recap

Lie group		Lie algebra		Enveloping algebra
$SL(2)$	$\rightsquigarrow$	$\mathfrak{sl}(2)$	$\rightsquigarrow$	$U(\mathfrak{sl}(2))$

We also defined an object $k[x, y]$ on which $U(\mathfrak{sl}(2))$ acts.

Before we "deform" $U(\mathfrak{sl}(2))$, we first introduce another algebraic structure fundamental to the study of quantum groups.

Example

Let k be a field and G be a finite group. G has an associative multiplicative map, an inverse map, and an identity e.

Consider the vector space of k-valued functions on G, which we will denote $C(G)$. $C(G)$ has the natural structure of an algebra under pointwise multiplication.
This algebra and, in fact, any algebra A can be characterized as a vector space equipped with two linear maps $\mu : A \otimes A \to A$ and $\eta : k \to A$ which satisfy the following conditions:

- (Associativity) The diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \tag{19}$$

  commutes. This means the multiplicative operation is associative.
- (Unit) The diagram:

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \tag{20}$$

  commutes. This means the element $\eta(1)$ is a left and right unit for μ.

Recall from last week, $C(G)$ also carries the maps:

- a comultiplication $\Delta : C(G) \longrightarrow C(G \times G)$:
$$\Delta(f) : (g_1, g_2) \longmapsto f(g_1 g_2)$$
- a counit $\varepsilon : C(G) \longrightarrow k$:
$$\varepsilon(f) = f(e)$$
- an antipode $S : C(G) \longrightarrow C(G)$:
$$S(f) : g \longmapsto f(g^{-1})$$

The maps Δ, ε, and S satisfy the following conditions for $C = C(G)$:

- (Coassociativity) The diagram:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C \otimes C \otimes C \end{array} \tag{21}$$

  commutes.

- (Counit) The diagram:

$$\begin{array}{ccccc} k \otimes C & \xleftarrow{\varepsilon\otimes\mathrm{id}} & C \otimes C & \xrightarrow{\mathrm{id}\otimes\varepsilon} & C \otimes k \\ & \nwarrow \cong & \uparrow{\scriptstyle \Delta} & \nearrow \cong & \\ & & C & & \end{array} \tag{22}$$

  commutes.

The maps Δ, ε, and S also satisfy:

- Coinvertibility:
$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \qquad f(g^{-1}g) = f(e) = f(gg^{-1})$$

- Δ and ε are morphisms of algebras.

This is enough to conclude $C(G)$ is what we will call a "Hopf algebra."

Hopf Algebras

Definition

A Hopf Algebra is a sextuple $(H, \mu, \eta, \Delta, \varepsilon, S)$ such that (H, μ, η) is an algebra, Δ and ε are algebra morphisms satisfying the conditions of coassociativity and counit, and S satisfies the condition of coinvertibility.

Example

$U(\mathfrak{sl}(2))$ is a Hopf algebra.

Consider the maps:

$$\begin{aligned} \delta :& \mathfrak{sl}(2) \to \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \\ & x \mapsto (x, x) \\ z :& \mathfrak{sl}(2) \to \{0\} \\ & x \mapsto 0 \\ \mathrm{op} :& \mathfrak{sl}(2) \to \mathfrak{sl}(2) \\ & x \mapsto -x \end{aligned} \tag{23}$$

Using corollaries of the universal property of enveloping algebras, we can use these maps to derive a coproduct $\Delta : U(\mathfrak{sl}(2)) \to U(\mathfrak{sl}(2)) \otimes U(\mathfrak{sl}(2))$, a counit $\varepsilon : U(\mathfrak{sl}(2)) \to \mathbb{C}$, and a antipode $S : U(\mathfrak{sl}(2)) \to U(\mathfrak{sl}(2))$.

A Coproduct on $U(\mathfrak{sl}(2))$

Proposition

Explicitly, the coproduct Δ on $U(\mathfrak{sl}(2))$ is given by:

$$\Delta(x_1 \cdots x_n) = 1 \otimes x_1 \cdots x_n + \sum_{p=1}^{n-1} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)} + x_1 \cdots x_n \otimes 1 \tag{24}$$

where σ runs over all (p, q)-shuffles of the symmetric group S_n.

Commutativity and Co-commutativity

Additionally, a Hopf algebra's product μ may be commutative and its coproduct Δ may be cocommutative. Let

$$\tau_{A,A}(a \otimes a') = a' \otimes a \tag{25}$$

- (Commutativity) The diagram:

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \tag{26}$$

  commutes.

- (Cocommutativity) The diagram:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\tau_{C,C}"] & & C \otimes C \end{tikzcd} \tag{27}$$

  commutes.

Example

$U(\mathfrak{sl}(2))$ is a cocommutative but not necessarily commutative Hopf algebra.

Now, we will construct a one parameter deformation of the enveloping algebra of $\mathfrak{sl}(2)$, $U(\mathfrak{sl}(2))$.

Definition

Let q be an invertible element of our field $\mathbb{C}$. Consider the algebra generated by the variables X, Y, H, and K under the relations:

$$\begin{gathered} KK^{-1} = K^{-1}K = 1, \\ KXK^{-1} = q^2X,\ KYK^{-1} = q^{-2}Y \\ [X, Y] = H,\ (q - q^{-1})H = K - K^{-1}, \\ [H, X] = q(XK + K^{-1}X),\ [H, Y] = -q^{-1}(YK + K^{-1}Y) \end{gathered} \tag{28}$$

This algebra will be denoted $U_q(\mathfrak{sl}(2))$ and is a q-deformation of the enveloping algebra $U(\mathfrak{sl}(2))$.

In particular, if we set $q = 1$ and $K = 1$, we get the relations:

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y \tag{29}$$

and get $U_1(\mathfrak{sl}(2))/(K - 1) = U(\mathfrak{sl}(2))$.

We will call $U_q(\mathfrak{sl}(2))$ a quantum group. $U_q(\mathfrak{sl}(2))$ is also another example of a Hopf algebra.

Proposition

Unlike $U(\mathfrak{sl}(2))$ which was cocommutative, $U_q(\mathfrak{sl}(2))$ is neither commutative nor cocommutative.

$$SL(2) \quad \rightsquigarrow \quad \mathfrak{sl}(2) \quad \rightsquigarrow \quad U(\mathfrak{sl}(2)) \quad \overset{q\text{-deformation}}{\rightsquigarrow} \quad U_q(\mathfrak{sl}(2))$$

Definition

Let q be an invertible element of the ground field k, and let I_q be the two-sided ideal of the free algebra $k\{x, y\}$ generated by the element $yx - qxy$. We define the *quantum plane* as the quotient-algebra

$$k_q[x, y] = k\{x, y\}/I_q \tag{30}$$

Now, let's assume the invertible element q is different than 1 and -1. This allows us to define:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{31}$$

for any integer n.

Definition

Now we define q-analogues of partial derivatives:

$$\frac{\partial_q(x^m y^n)}{\partial x} = [m]x^{m-1}y^n \qquad \frac{\partial_q(x^m y^n)}{\partial y} = [n]x^m y^{n-1} \tag{32}$$

along with two useful functions (automorphisms) on the quantum plane σ_x, σ_y:

$$\sigma_x(x) = qx, \qquad \sigma_x(y) = y, \qquad \sigma_y(x) = x, \qquad \sigma_y(y) = qy \tag{33}$$

Proposition

We have $k[x,y]$ is a $U(\mathfrak{sl}(2))$ module with the action on the polynomial algebra $k[x,y]$ given by:

$$XP = x\frac{\partial P}{\partial y} \qquad YP = y\frac{\partial P}{\partial x} \qquad HP = x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y} \tag{34}$$

where P is some polynomial in $k[x,y]$.

In other words, we have an algebra homomorphism $\rho : U(\mathfrak{sl}(2)) \to \mathrm{End}(k[x,y])$. This homomorphism is defined on the basis:

$$\rho(X) = x\frac{\partial}{\partial y} \qquad \rho(Y) = y\frac{\partial}{\partial x} \qquad \rho(H) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{35}$$

Proposition

We have $k_q[x,y]$ is a $U_q(\mathfrak{sl}(2))$ module with the action on the polynomial algebra $k_q[x,y]$ given by:

$$\begin{gathered} XP = x\frac{\partial_q P}{\partial y} \qquad YP = \frac{\partial_q P}{\partial x}y \qquad KP = (\sigma_x\sigma_y^{-1})(P) \\ HP = \frac{1}{q-q^{-1}}(\sigma_x\sigma_y^{-1} - \sigma_y\sigma_x^{-1})(P) \end{gathered} \tag{36}$$

where P is some polynomial in $k_q[x,y]$.

In other words, we have an algebra homomorphism $\rho : U_q(\mathfrak{sl}(2)) \to \mathrm{End}(k_q[x,y])$. This homomorphism is defined on the basis:

$$\begin{gathered} \rho(X) = x\frac{\partial_q}{\partial y} \qquad \rho(Y) = \frac{\partial_q}{\partial x}y \qquad \rho(K) = (\sigma_x\sigma_y^{-1}) \\ \rho(H) = \frac{1}{q-q^{-1}}(\sigma_x\sigma_y^{-1} - \sigma_y\sigma_x^{-1}) \end{gathered} \tag{37}$$

A Final Comment

$$SL(2) \quad \rightsquigarrow \quad \mathfrak{sl}(2) \quad \rightsquigarrow \quad U(\mathfrak{sl}(2)) \quad \overset{q\text{-deformation}}{\rightsquigarrow} \quad U_q(\mathfrak{sl}(2))$$

$$k[x,y] \quad \overset{q\text{-deformation}}{\rightsquigarrow} \quad k_q[x,y]$$

The fact that $U(\mathfrak{sl}(2))$'s action on the affine plane has a quantum analogue is just one example of the way in which the (very well developed) representation theory of $U(\mathfrak{sl}(2))$ translates nicely into the quantum case.

Thanks for listening!

Sources:

Quantum Groups, by Christian Kassel
Lie Algebras, by F. Gonzalez
Introduction to Lie Groups and Lie Algebras, by A. Kirillov Jr.

Proposition

Any two irreducible $U(\mathfrak{sl}(2))$-module $V(n)$ of dimension $n+1$ are isomorphic. In particular, $V(n) \cong k[x, y]_n$, a submodule of the affine plane containing homogenous polynomials of degree n.

Proposition

(Clebsch-Gordan Formula) Consider two nonnegative integers $n \geq m$ and let $V(n)$ and $V(m)$ be a $U(\mathfrak{sl}(2))$-modules of dimension $n+1$ and $m+1$ respectively. Then there exists an isomorphism of $U(\mathfrak{sl}(2))$-modules:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m) \tag{38}$$

Assume q is not a root of unity.

Proposition

Any two irreducible $U_q(\mathfrak{sl}(2))$-module $V(n)$ of dimension $n+1$ are isomorphic. In particular, $V(n) \cong k_q[x,y]_n$, a submodule of the affine plane containing homogenous polynomials of degree n.

Proposition

(Quantum Clebsch-Gordan Formula) Consider two nonnegative integers $n \geq m$ and let $V(n)$ and $V(m)$ be a $U_q(\mathfrak{sl}(2))$-modules of dimension $n+1$ and $m+1$ respectively. Then there exists an isomorphism of $U_q(\mathfrak{sl}(2))$-modules:

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m+2) \oplus V(n-m) \tag{39}$$