Introduction to Quantum Groups

I. Groups, Algebras and Duality

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Next on *Introduction to Quantum Groups*

- Oct. 7: *q-Deformations of Lie algebras*
  E. Shelburne'21
- Oct. 21: *Quantum automorphisms of graphs*
  S. Phillips'21 and A. Pisharody'21
- Oct. 28: *TBD*
  M. Weber (Saarland University)
- Nov. 11: *TBD*
  E. Swartz

Shocking Revelation

Noncommutative Geometry is *pointless* and quantum groups don't really exist.

Oversimplification

Quantum objects are given in terms of algebraic structures obtained by altering structures associated with classical objects.

Outline of future talks

Lie group		Lie algebra		Enveloping algebra		Quantum group
G	$\rightsquigarrow$	$\mathfrak{g}$	$\rightsquigarrow$	$U(\mathfrak{g})$	$\overset{q\text{-deformation}}{\rightsquigarrow}$	$U_q(\mathfrak{g})$
Graph		Automorphisms		C*-algebra		Quantum automorphisms
Γ	$\rightsquigarrow$	$\mathrm{Aut}(\Gamma)$	$\rightsquigarrow$	$C(\mathrm{Aut}(\Gamma))$	$\overset{\text{liberation}}{\rightsquigarrow}$	$C(\mathrm{Aut}^+(\Gamma))$

Today's objectives

1. C*-algebras are the right language for quantum (noncommutative) spaces.

2. Generic instance of liberation: quantum groups à la Woronowicz

Compact group		Commutative C*-algebra		Woronowicz algebra
G	$\rightsquigarrow$	$C(G)$	$\rightsquigarrow$	$\mathcal{A}$
				(Matrix quantum group)

Let X be a locally compact (Hausdorff) space. No algebraic structure.

Consider:

$$C_0(X) = \left\{f : X \longrightarrow \mathbb{C}\,,\ \text{continuous with } \lim_{x\to\infty} f(x) = 0\right\}.$$

Then $C_0(X)$ is an algebra over $\mathbb{C}$:

$$(\lambda f + g)(x) = \lambda f(x) + g(x) \qquad , \qquad (fg)(x) = f(x)g(x).$$

It also carries a norm:

$$||f|| = \sup\left\{|f(x)|\,,\ x \in X\right\}$$

for which it is complete, and an involution

$$f^*(x) = \overline{f(x)}$$

such that

$$||f^* f|| = ||f||^2.$$

In other words, $C_0(X)$ is a C*-algebra.

Remark: multiplication in $C_0(X)$ is commutative.

Topological properties of X can be read off $C_0(X)$.

For instance:

Theorem

X is compact $\quad\Leftrightarrow\quad$ $C_0(X)$ has a multiplicative identity.

Proof: a multiplicative identity e of $C_0(X)$ must satisfy

$$e(x)f(x) = f(x)$$

for all $x \in X$ and every $f \in C_0(X)$. Necessarily, $e(x) = 1$ for all $x \in X$.
Constant functions are continuous and $\lim_{x\to\infty} 1 = 0$ if and only if X is compact $\square$

Points in X give continuous linear functionals on $C_0(X)$.

For $x \in X$, consider

$$\begin{array}{rccc} \mathrm{eval}_x : & C_0(X) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(x) \end{array}.$$

Every (locally compact) space X gives rise to a C*-algebra $C_0(X)$.

What C*-algebras are obtained in this way?

Obviously, only *commutative* C*-algebras. All of them?

Theorem (Gelfand-Naimark-Segal, 1943)

Every C^*-algebra A is isomorphic to a Closed $*$-invariant subalgebra of continuous linear operators on some Hilbert space $\mathcal{H}$:

$$A \subset B(\mathcal{H}).$$

Theorem (Gelfand-Naimark, 1943)

If A is a commutative C^*-algebra, there exists a locally compact Hausdorff space X (the spectrum of A) such that

$$A \simeq C_0(X).$$

Classical spaces $\longleftrightarrow$ Commutative C*-algebras.

Let $\theta \in \mathbb{R}$. Consider a unital C*-algebra A_θ generated by two elements U and V such that

$$U^* U = 1 \quad , \quad V^* V = 1 \quad , \quad UV = e^{2i\pi\theta} VU.$$

If $\theta = 0$ then A_θ is commutative, hence an algebra of functions by Gelfand-Naimark's theorem. In fact,

$$A_0 \simeq C(\mathbb{T}^2)$$

where $\mathbb{T}^2 = S^1 \times S^1$ is the 2-torus.

In general, A_θ is not commutative, so *not* an algebra of functions... but we still write

$$\boxed{A_\theta = C(\mathcal{T}^2_\theta)}$$

and say that $\mathcal{T}^2_\theta$ is a noncommutative torus.

Just because a space doesn't exist, doesn't mean it can't have nice properties.

Theorem. $\mathcal{T}^2_\theta$ is compact.

Proof (interpretation): A_θ contains a multiplicative unit. □

Remark: $\mathcal{T}^2_\theta$ may not have points but A_θ has linear functionals, representations...

Gelfand theory says:

$$\begin{array}{ccc} \text{Classical spaces} & \stackrel{\sim}{\longleftrightarrow} & \text{Commutative C*-algebras} \\ X & \longmapsto & C_0(X) \end{array}$$

Relaxing the commutativity condition,

$$\begin{array}{ccc} \text{Quantum spaces} & \longleftrightarrow & \text{(Possibly) noncommutative C*-algebras} \\ & \longmapsto & A \end{array}$$

What is a quantum group?

Strategy:

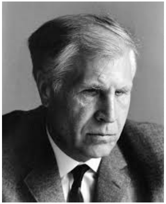
1. Associate algebras to ordinary groups. $k[G]$, $C_r^*(G)$, $\mathcal{L}(G)$, $C(G)$, $\mathfrak{g}$, $U(\mathfrak{g})$...
2. Characterize them.
3. Relax conditions to get new objects (and call them quantum groups).

What algebras should we choose?

Let G be a locally compact abelian group: $\mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{T}$, $\mathbb{R}$, $\mathbb{Q}_p$, ...

The *Pontryagin dual* of G is

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{T}) = \left\{ \chi : G \longrightarrow \mathbb{C}, \text{ cont. hom. with } |\chi(g)| \equiv 1 \right\}$$



Lev Pontryagin
1908 - 1988

Theorem (Pontryagin Duality)

$\widehat{G}$ is a locally compact abelian group and $\widehat{\widehat{G}} \simeq G$.

Examples of characters of $\mathbb{T}$ and $\mathbb{R}$:

$$\chi_n(z) = z^n \qquad \text{and} \qquad \chi_\xi(x) = e^{ix\xi}$$

- $\widehat{\mathbb{R}} = \left\{ \chi_\xi \,,\, \xi \in \mathbb{R} \right\} \simeq \mathbb{R}$
- $\widehat{\mathbb{T}} = \left\{ \chi_n \,,\, n \in \mathbb{Z} \right\} \simeq \mathbb{Z}$ and by Pontryagin duality, $\widehat{\mathbb{Z}} \simeq \mathbb{T}$.

Theorem

The Pontryagin dual of a compact (resp. discrete) abelian group is discrete (resp. compact.)

Now, back to the main question...

What algebra related to a group G should we study (and then deform)?

- $C_0(G)$: functions on G with pointwise multiplication
  - C*-algebra, always commutative,
  - unital if and only if G is compact (then $C_0(G) = C(G)$).
- $C_r^*(G)$: (completion of) $C_c(G)$ with convolution (= $\mathbb{C}[G]$ if G is finite)
  - C*-algebra, commutative if and only if G is abelian,
  - unital if and only if G is discrete.

Theorem (Fourier Analysis on Abelian Groups)

Let G be an abelian group. For $f \in C_c(G)$ and $\chi \in \widehat{G}$, let

$$\widehat{f}(\chi) = \int_G f(g)\overline{\chi(g)}\, dg.$$

Then $\widehat{f} \in C_0(\widehat{G})$ and the Fourier transformation $f \mapsto \widehat{f}$ extends to an isomorphism

$$C_r^*(G) \simeq C_0(\widehat{G}).$$

If G is abelian, $C_0(G)$ and $C_r^*(G)$ are dual choices.

If G is non-abelian and non-compact, $\widehat{G}$ is not a group...

Goal for the rest of the talk

Describe a framework that accommodates algebras such as $C_0(G)$ and $C_r^*(G)$, in which Fourier-Pontryagin duality is restored.

Neither G nor $\widehat{G}$ will really exist, but they will be dual to each other. 

First, let us characterize $C_0(G)$ when G is a classical group.

Two questions

- How does the algebraic structure of G manifest at the level of $C_0(G)$?
- Can $C_0(G)$ be described by generators and relations?

Let G be a group with identity e. The algebra $C(G)$ carries:

- a comultiplication $\Delta : C(G) \longrightarrow C(G \times G)$:
$$\Delta(f) : (g_1, g_2) \longmapsto f(g_1 g_2)$$
- a counit $\varepsilon : C(G) \longrightarrow \mathbb{C}$:
$$\varepsilon(f) = f(e)$$
- an antipode $S : C(G) \longrightarrow C(G)$:
$$S(f) : g \longmapsto f(g^{-1})$$

Since G is a group, Δ, ε and S satisfy:

- Coassociativity:
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad f((g_1 g_2) g_3) = f(g_1 (g_2 g_3))$$
- Counitality:
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta \qquad f(e\,\cdot) = f = f(\cdot\, e)$$
- Coinvertibility:
$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \qquad f(g^{-1} g) = f(e) = f(g g^{-1})$$

The algebraic structure of G is encoded in $\big(C(G), \Delta, \varepsilon, \eta, S\big)$.

Now assume that G is a compact (Lie) group (of unitary matrices):

$$G \subset \mathrm{U}(n).$$

Consider the elementary matrix coefficient functions $u_{ij} \in C(G)$:

$$u_{ij} : \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ & \ddots & \\ \vdots & g_{ij} & \vdots \\ & & \ddots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \longmapsto g_{ij}$$

Facts:

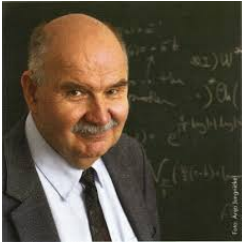
- The functions $\{u_{ij}\ ,\ 1 \leq i, j \leq n\}$ generate the C*-algebra $C(G)$.
- They form a biunitary matrix $u = [u_{ij}]$ in $M_n(C(G))$.
- $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. It is a morphism $\Delta : C(G) \longrightarrow C(G) \otimes C(G)$.
- $\varepsilon(u_{ij}) = \delta_{ij}$. It is a morphism $\varepsilon : C(G) \longrightarrow \mathbb{C}$.
- $S(u_{ij}) = \overline{u_{ji}} = u_{ji}^*$ defines a morphism $S : C(G) \longrightarrow C(G)^{\mathrm{op}}$.

Woronowicz algebras

~~Characterization of $C(G)$~~ **Definition** (Woronowicz, 1987)

A *matrix quantum group* is a unital C*-algebra $\mathcal{A}$ with maps Δ, ε and S such that:

- $\mathcal{A}$ is generated by n^2 elements $\{u_{ij}\ ,\ 1 \leq i, j \leq n\}$.
- They form a biunitary matrix $u = [u_{ij}]$ in $M_n(\mathcal{A})$.
- $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : \mathcal{A} \longrightarrow \mathbb{C}$.
- $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{op}}$.



Recall the correspondence given by Gelfand theory:

$$\begin{array}{ccc} \text{Locally compact spaces} & \overset{\sim}{\longleftrightarrow} & \text{Commutative C*-algebras} \\ X & \longmapsto & C_0(X) \end{array}$$

Now we have:

$$\begin{array}{ccc} \text{Compact groups} & \longrightarrow & \text{Commutative Wornowicz algebras} \\ G & \longmapsto & C(G) \end{array}$$

$\rightsquigarrow$ Is this a one-to-one correspondence?

Theorem

Every commutative Woronowicz algebra is of the form $C(G)$, where G is a compact group.

<u>Proof</u>: let $\mathcal{A}$ be a commutative Woronowicz algebra. Since $\mathcal{A}$ is unital, it has compact spectrum. Let $G = \mathrm{Sp}(\mathcal{A})$ and consider the embedding

$$\mathrm{Sp}(\mathcal{A}) \ni \varphi \longmapsto [\varphi(u_{ij})] \in \mathrm{U}(n).$$

Properties of Δ, ε and S show that the image of this embedding is a subgroup of $\mathrm{U}(n)$. $\square$

Are there interesting examples of noncommutative Woronowicz algebras?

Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be a finitely generated group and consider the associated C*-algebra:

$$C^*(\Gamma) \quad \overset{\text{dense}}{\supset} \quad \mathbb{C}[\Gamma] \quad \supset \quad \Gamma.$$

It is a Woronowicz algebra with defining matrix:

$$u = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \end{bmatrix} \in M_n\left(C^*(\Gamma)\right)$$

and structure maps defined by:

$$\Delta(\gamma) = \gamma \otimes \gamma \qquad , \qquad \varepsilon(\gamma) = 1 \qquad , \qquad S(\gamma) = \gamma^{-1}.$$

Theorem

All co-commutative Woronowicz algebras are of the form $C^*(\Gamma)$.

General picture

Let $\mathcal{A}$ be a Woronowicz algebra.

- If $\mathcal{A}$ is commutative, it is of the form $C(G)$ with G compact group.
- If $\mathcal{A}$ is co-commutative, it is of the form $C^*(\Gamma)$ with Γ discrete group.

With this in mind, if $\mathcal{A}$ is an arbitrary Woronowicz algebra, we write

$$\mathcal{A} = C(G) = C^*(\Gamma)$$

and call G and Γ quantum groups.

Remarks:

- This generalizes Pontryagin duality: the dual of a compact (resp. discrete) quantum group is a discrete (resp. compact) quantum group.
- The dense $*$-algebra of $\mathcal{A}$ generated by the $u'_{ij}s$ generalizes $C^\infty(G)$ if G is compact and $\mathbb{C}[\Gamma]$ if Γ is discrete.

Thank you.